HYDRODYNAMIC RESISTANCE OF PARTICLES AT SMALL REYNOLDS NUMBERS

Howard Brenner*

Department of Chemical Engineering
New York University, Bronx, New York

I. Introduction

The forces acting on a volume element in a fluid continuum are essentially of three types: (i) viscous forces, expressing resistance to change of shape; (ii) inertial forces, expressing resistance to changes in speed and direction; (iii) external volume forces—typically gravitational or electrodynamic in origin. The subject matter of the present review is concerned with situations where inertial forces are small compared with viscous forces, at least in a

* Present address: Department of Chemical Engineering, Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

global sense. In short, we deal with low Reynolds number flows—more specifically with the hydrodynamic resistance of particles in the Stokes and Oseen regimes. The distinguishing feature of such flows is the linearity[1] or quasi-linearity of the underlying equations of motion. This is in marked contrast to laminar boundary layer flows, which lie at the opposite extreme of the Reynolds number spectrum.

In order to keep the size of the review within moderate bounds while, at the same time, furnishing sufficient detail to make this summary intelligible to the nonspecialist, we have—somewhat arbitrarily—eliminated a number of important topics which would ordinarily have found a place here. Active areas of contemporary research pertaining directly to Stokes and Oseen flows around particles that are *not* systematically discussed here include the following: two-dimensional flows; unsteady flows; deformable particles, including liquid drops; non-Newtonian flows; magnetohydrodynamic flows; compressible flows; numerical and variational mathematical methods in low Reynolds number flows; and heat and mass transfer from particles. Brief comments on some of these topics are, however, offered where pertinent to the main discussion. This partial listing furnishes some indication of the remarkable growth of the field during recent years. Readers interested in a broader and more systematic survey of the general area of low Reynolds number flows are directed to the following recent monographs and review articles: B4, F13, H9, H11, I1, L1, L4, L4a, L9, M8. The classical treatises of Oseen (O4), Villat (V4), and Lamb (L5), though somewhat dated, are still serviceable. Landau and Lifshitz's text (L6, Chapter II) provides a concise, but illuminating, summary from an elementary viewpoint.

The recent development of tensorial schemes for characterizing the intrinsic hydrodynamic resistance of particles of arbitrary shape, and the application of singular perturbation techniques to obtain asymptotic solutions of the Navier–Stokes equations at small Reynolds numbers constitute significant contributions to our understanding of "slow" viscous flow around bodies. It is with these topics that this review is primarily concerned. In presenting this material we have elected to use Gibbs' polyadics in preference to conventional tensor notation. For in our view, the former symbolism—dealing as it does with *direction* as a primitive concept—is more closely related to the physical world in which we live than is the latter notation.

[1] Not all such flows, however, are linear—as, for example, in the case of *non-Newtonian* creeping flows around spherical particles (B4a, B4b, C1, D3, F9, F10, G5, L8, L10, R1, S2, S10, T4, T7, W2, W3, W3a, W3b, W4, W5, W6, Z1). Similarly, owing to the unknown shape of the interface at the outset, free-boundary problems involving liquid droplets in nonuniform flows (Section II, C, 2, *b*) are intrinsically nonlinear despite the possible linearity of the equations of motion (and boundary conditions) inside and outside of the droplet.

Moreover, polyadic symbols, being free from extraneous indices which detract from the invariant significance of the *single* entities they represent, are more suggestive and possess greater aesthetic appeal than do their tensor counterparts. Our polyadic notation is essentially equivalent to that employed in Milne's book (M16). Other useful discussions of this notation will be found in References B6, D6, G3, M15.

II. Stokes Flows

A. Introduction

A nonspherical particle is generally anisotropic with respect to its hydrodynamic resistance; that is, its resistance depends upon its orientation relative to its direction of motion through the fluid. A complete investigation of particle resistance would therefore seem to require experimental data or theoretical analysis for each of the infinitely many relative orientations possible. It turns out, however, at least at small Reynolds numbers, that particle resistance has a tensorial character and, hence, that the resistance of a solid particle of any shape can be represented for all orientations by a few tensors. And the components of these tensors can be determined from either theoretical or experimental knowledge of the resistance of the particle for a *finite* number of relative orientations. The tensors themselves are intrinsic geometric properties of the particle alone, depending only on its size and shape. These observations and various generalizations thereof furnish most, but not all, of the subject matter of this section.

B. Equations of Motion

The Navier–Stokes and continuity equations for an incompressible, nonpolar (D1, D2), Newtonian fluid are

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \nabla^2 \mathbf{v} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

where ρ and μ are the fluid density and viscosity (both assumed constant), $\mathbf{v}$ and p are the local fluid velocity and pressure, and t is the time. The stress dyadic for an incompressible Newtonian fluid is

$$\boldsymbol{\pi} = -\mathbf{I}p + \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}] \tag{3}$$

In Eq. (1) it is assumed that the only external body forces acting on the fluid are either due to gravity or are otherwise derivable from a potential. As such, these forces have been absorbed into the pressure term.

Let a, V, and τ be, respectively, a representative particle length, speed, and time, and let $R = aV\rho/\mu$ be the particle Reynolds number. By introducing the dimensionless variables

$$\underline{\mathbf{v}} = \frac{\mathbf{v}}{V} \qquad \underline{p} = \frac{pa}{\mu V} \qquad \underline{t} = \frac{t}{\tau}, \qquad \underline{\mathbf{r}} = \frac{\mathbf{r}}{a} \qquad \underline{\nabla} = a\nabla \tag{4}$$

where $\mathbf{r}$ is the position vector, Eqs. (1) and (2) may be written in the dimensionless forms

$$R\left(\frac{a}{\tau V}\frac{\partial \underline{\mathbf{v}}}{\partial \underline{t}} + \underline{\mathbf{v}} \cdot \underline{\nabla}\underline{\mathbf{v}}\right) = -\underline{\nabla}\underline{p} + \underline{\nabla}^2\underline{\mathbf{v}} \tag{5}$$

$$\underline{\nabla} \cdot \underline{\mathbf{v}} = 0 \tag{6}$$

If the motion is steady when viewed from an inertial reference frame, the term $\partial\mathbf{v}/\partial t$ vanishes identically. Providing that the remaining nondimensional terms remain finite as $R \to 0$, these then reduce to Stokes equations at sufficiently small Reynolds numbers. In dimensional form, Stokes equations are

$$\nabla^2\mathbf{v} = \frac{1}{\mu}\nabla p \tag{7}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{8}$$

These also result even if the motion is unsteady, providing that $a/\tau V$ and the other dimensionless terms remain finite in the limit $R = 0$.[2] Equations (7) and (8) are then referred to as the *quasi-static* or *quasi-steady* Stokes equations. In this case the time variable enters the equations of motion only in an implicit form. The precise relationship between the solutions of Eqs. (7) and (8) and the asymptotic solutions of the Navier–Stokes equations at small Reynolds numbers is discussed in Section III.

C. Single Particles

1. *Quiescent Fluids*

In this section we treat the steady and quasi-steady motions of rigid, three-dimensional particles in a fluid at rest at infinity on the basis of Stokes

[2] See, for example, the discussions (I2, K2) of unsteady low Reynolds number flows based on singular perturbation methods.

A general discussion of the circumstances in which it is permissible to adopt the quasi-static approximation is provided by Happel and Brenner (H9, pp. 52–55).

equations (B18, B19, B22). Since the fluid adheres to the surface of the solid particle the appropriate boundary conditions are

$$\mathbf{v} = \mathbf{U}_O + \boldsymbol{\omega} \times \mathbf{r}_O \qquad \text{on the body } B \tag{9}$$

$$\mathbf{v} \to \mathbf{0} \qquad \text{at infinity} \tag{10}$$

where O may be any point permanently affixed to the particle. $\mathbf{U}_O$ is the instantaneous translational velocity of point O and $\boldsymbol{\omega}$ is the instantaneous angular velocity of the particle. The instantaneous axis of rotation need *not* pass through O. The position vector $\mathbf{r}_O$ is drawn from an origin at O.

Because of linearity, the translational and rotational motions may be separately treated. Thus, one may define a translational Stokes flow $\overset{(t)}{\mathbf{v}}_O, \overset{(t)}{p}_O$ satisfying Eqs. (7) and (8) and the boundary conditions

$$\overset{(t)}{\mathbf{v}}_O = \mathbf{U}_O \qquad \text{on } B \tag{11}$$

$$\overset{(t)}{\mathbf{v}}_O \to \mathbf{0} \qquad \text{at infinity} \tag{12}$$

The subscript O indicates that the field depends on the choice of origin. Similarly, one may define a rotational Stokes field satisfying the boundary conditions

$$\overset{(r)}{\mathbf{v}}_O = \boldsymbol{\omega} \times \mathbf{r}_O \qquad \text{on } B \tag{13}$$

$$\overset{(r)}{\mathbf{v}}_O \to \mathbf{0} \qquad \text{at infinity} \tag{14}$$

The general field satisfying (9) and (10) is then

$$\mathbf{v} = \overset{(t)}{\mathbf{v}}_O + \overset{(r)}{\mathbf{v}}_O \tag{15}$$

$$p = \overset{(t)}{p}_O + \overset{(r)}{p}_O \tag{16}$$

Both the translational and rotational fields depend upon the viscosity of the fluid and the magnitude and orientation of the vectors $\mathbf{U}_O$ and $\boldsymbol{\omega}$ relative to axes fixed in the body. By virtue of linearity, however, it is possible to define dyadic "velocity" fields and vector "pressure" fields which are wholly independent of these parameters. Thus, one may define a translational dyadic velocity field $\overset{(t)}{\mathbf{V}}$ and its associated vector pressure field $\overset{(t)}{\mathbf{P}}$ by the differential equations (B22)

$$\nabla^2\overset{(t)}{\mathbf{V}} - \nabla\overset{(t)}{\mathbf{P}} = \mathbf{0} \tag{17}$$

$$\nabla \cdot \overset{(t)}{\mathbf{V}} = \mathbf{0} \tag{18}$$

and boundary conditions

$$\overset{(t)}{\mathbf{V}} = \mathbf{I} \qquad \text{on } B \tag{19}$$

$$\overset{(t)}{\mathbf{V}} \to \mathbf{0} \qquad \text{at infinity} \tag{20}$$

where **I** is the dyadic idemfactor. The ordinary translational velocity and pressure fields are then expressible in terms of these higher-order fields via the relations

$$\overset{(t)}{\mathbf{v}}_O = \overset{(t)}{\mathbf{V}} \cdot \mathbf{U}_O \tag{21}$$

$$\overset{(t)}{p}_O = \mu\overset{(t)}{\mathbf{P}} \cdot \mathbf{U}_O \tag{22}$$

for all values of μ, $\mathbf{U}_O$, and all choices of origin O. The fields $\overset{(t)}{\mathbf{V}}$ and $\overset{(t)}{\mathbf{P}}$ are themselves independent of these parameters. By means of Eqs. (21) and (22), $\overset{(t)}{\mathbf{V}}$ and $\overset{(t)}{\mathbf{P}}$ may be determined from knowledge of $\overset{(t)}{\mathbf{v}}_O$ and $\overset{(t)}{p}_O$ for any three noncoplanar values of $\mathbf{U}_O$. Alternatively, they can be obtained directly by solving the dyadic equations of motion, (17) to (20).

A similar analysis applies to the rotational motion. In particular (B22)

$$\overset{(r)}{\mathbf{v}}_O = \overset{(r)}{\mathbf{V}}_O \cdot \boldsymbol{\omega} \tag{23}$$

$$\overset{(r)}{p}_O = \mu\overset{(r)}{\mathbf{P}}_O \cdot \boldsymbol{\omega} \tag{24}$$

where the rotational dyadic velocity field $\overset{(r)}{\mathbf{V}}_O$ and vector pressure field $\overset{(r)}{\mathbf{P}}_O$ are the solutions of the equations

$$\nabla^2\overset{(r)}{\mathbf{V}}_O - \nabla\overset{(r)}{\mathbf{P}}_O = \mathbf{0} \tag{25}$$

$$\nabla \cdot \overset{(r)}{\mathbf{V}}_O = \mathbf{0} \tag{26}$$

$$\overset{(r)}{\mathbf{V}}_O = \boldsymbol{\varepsilon} \cdot \mathbf{r}_O \qquad \text{on } B \tag{27}$$

$$\overset{(r)}{\mathbf{V}}_O \to \mathbf{0} \qquad \text{at infinity} \tag{28}$$

in which $\boldsymbol{\varepsilon}$ is the unit isotropic triadic.[3] These polyadic fields, though independent of μ and $\boldsymbol{\omega}$, do depend upon the location of O, as required by the right-hand member of Eq. (27).

As examples of these intrinsic translational and rotational velocity fields we note that one can express Stokes' original solution for a translating sphere of radius a (L5, p. 597) in the form

$$\overset{(t)}{\mathbf{V}} = \frac{3}{4}\frac{a}{r}\left(\mathbf{I} + \frac{\mathbf{rr}}{r^2}\right) + \frac{1}{4}\left(\frac{a}{r}\right)^3\left(\mathbf{I} - 3\frac{\mathbf{rr}}{r^2}\right) \tag{29}$$

$$\overset{(t)}{\mathbf{P}} = \frac{3a}{2}\frac{\mathbf{r}}{r^3} \tag{30}$$

Similarly, if O is chosen at the sphere center, the solution for a rotating sphere (L5, p. 588) may be written as

$$\overset{(r)}{\mathbf{V}}_O = (a/r)^3\boldsymbol{\varepsilon}\cdot\mathbf{r} \tag{31}$$

$$\overset{(r)}{\mathbf{P}}_O = \mathbf{0} \tag{32}$$

where $\mathbf{r}$ is measured from O.

Equation (3) for the stress dyadic, applied to the translational motion, gives

$$\overset{(t)}{\boldsymbol{\pi}}_O = -\mathbf{I}\overset{(t)}{p}_O + \mu\left[\nabla\overset{(t)}{\mathbf{v}}_O + \left(\nabla\overset{(t)}{\mathbf{v}}_O\right)^\dagger\right] \tag{33}$$

[3] In terms of any right-handed triad of mutually perpendicular unit vectors $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ the unit isotropic triadic may be written as (M16)

$$\boldsymbol{\varepsilon} = \mathbf{i}_j\mathbf{i}_k\mathbf{i}_l\varepsilon_{jkl} \qquad \text{(summation convention)}$$

where ε_{jkl} is the permutation symbol

$$\varepsilon_{jkl} = \begin{cases} 0, & \text{if any two of } j, k, l \text{ are the same;} \\ 1, & \text{if } jkl\text{, in that order, is an even permutation of 1, 2, 3;} \\ -1, & \text{if } jkl\text{, in that order, is an odd permutation of 1, 2, 3} \end{cases}$$

Thus

$$\boldsymbol{\varepsilon} = \mathbf{i}_1\mathbf{i}_2\mathbf{i}_3 - \mathbf{i}_1\mathbf{i}_3\mathbf{i}_2 + \mathbf{i}_2\mathbf{i}_3\mathbf{i}_1 - \mathbf{i}_2\mathbf{i}_1\mathbf{i}_3 + \mathbf{i}_3\mathbf{i}_1\mathbf{i}_2 - \mathbf{i}_3\mathbf{i}_2\mathbf{i}_1$$

This invariant can be shown to be equivalent to

$$\boldsymbol{\varepsilon} = -\mathbf{I}\times\mathbf{I}$$

It thus follows from Eqs. (21) and (22) that there exists an intrinsic translational triadic "stress" field[4]

$$\overset{(t)}{\mathbf{\Pi}} = -\mathbf{I}\overset{(t)}{P} + \nabla\overset{(t)}{\mathbf{V}} + {}^{\dagger}\left(\nabla\overset{(t)}{\mathbf{V}}\right) \tag{34}$$

which is independent of the choice of μ, $\mathbf{U}_O$, and O, such that

$$\overset{(t)}{\pi}_O = \mu\overset{(t)}{\mathbf{\Pi}} \cdot \mathbf{U}_O \tag{35}$$

Analogously, for the rotational motion one has

$$\overset{(r)}{\pi}_O = \mu\overset{(r)}{\mathbf{\Pi}}_O \cdot \boldsymbol{\omega} \tag{36}$$

where

$$\overset{(r)}{\mathbf{\Pi}}_O = -\mathbf{I}\overset{(r)}{P}_O + \nabla\overset{(r)}{\mathbf{V}}_O + {}^{\dagger}\left(\nabla\overset{(r)}{\mathbf{V}}_O\right) \tag{37}$$

These intrinsic stress fields play fundamental roles in the theory of the hydrodynamic resistance of particles in the Stokes regime.

The hydrodynamic force $\mathbf{F}$ and torque about O, $\mathbf{T}_O$, exerted by the fluid on the particle (exclusive of buoyant forces and torques) are linear vector functions of the velocity and spin of the particle. In particular (B22)

$$\mathbf{F} = -\mu\left(\overset{(t)}{\mathbf{K}} \cdot \mathbf{U}_O + \overset{(c)}{\mathbf{K}}_O{}^{\dagger} \cdot \boldsymbol{\omega}\right) \tag{38}$$

$$\mathbf{T}_O = -\mu\left(\overset{(c)}{\mathbf{K}}_O \cdot \mathbf{U}_O + \overset{(r)}{\mathbf{K}}_O \cdot \boldsymbol{\omega}\right) \tag{39}$$

where $\overset{(t)}{\mathbf{K}}$ is the translation dyadic and $\overset{(c)}{\mathbf{K}}_O$ and $\overset{(r)}{\mathbf{K}}_O$ are, respectively, the coupling and rotation dyadics at O. These intrinsic hydrodynamic resistance coefficients are constant relative to axes fixed in the particle. They depend only upon the external geometric configuration of the particle wetted by the fluid; that is, on its size and shape. The two "direct" dyadics are related to the intrinsic stress fields via the expressions (B22)

$$\overset{(t)}{\mathbf{K}} = -\int_B d\mathbf{S} \cdot \overset{(t)}{\mathbf{\Pi}} \tag{40}$$

$$\overset{(r)}{\mathbf{K}}_O = -\int_B \mathbf{r}_O \times \left(d\mathbf{S} \cdot \overset{(r)}{\mathbf{\Pi}}_O\right) \tag{41}$$

[4] With regard to transposes of polyads and polyadics, we adopt the convention that the transposition operator (†) has the following properties:

$$(\cdots \mathbf{abcd})^{\dagger} = \cdots \mathbf{abdc}$$

and

$${}^{\dagger}(\mathbf{abcd} \cdots) = \mathbf{bacd} \cdots$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, ..., are any vectors. The former represents a *post*transposition, while the latter represents a *pre*transposition. For dyads and dyadics these operations are equivalent.

The "indirect," cross-term may be computed from either of the following formulas:

$$\overset{(c)}{\mathbf{K}}_O = -\int_B \mathbf{r}_O \times \left(d\mathbf{S} \cdot \overset{(t)}{\mathbf{\Pi}}\right) \tag{42a}$$

or

$$\overset{(c)}{\mathbf{K}}_O = -\int_B d\mathbf{S} \cdot \overset{(r)}{\mathbf{\Pi}}_O{}^{\dagger} \tag{42b}$$

the result being the same in either case. Here, $d\mathbf{S}$ is a directed element of surface area pointing into the fluid. Integration is over the wetted surface B of the body.

The dimensions of the translation, coupling, and rotation dyadics are, respectively, L, L^2, and L^3 ($L =$ length). Thus, if a be some characteristic length of the body, one may write Eqs. (38) and (39) in the forms

$$\mathbf{F} = -\mu\left(a\overset{(t)}{\underline{\mathbf{K}}} \cdot \mathbf{U}_O + a^2\overset{(c)}{\underline{\mathbf{K}}}_O{}^{\dagger} \cdot \boldsymbol{\omega}\right)$$

$$\mathbf{T}_O = -\mu\left(a^2\overset{(c)}{\underline{\mathbf{K}}}_O \cdot \mathbf{U}_O + a^3\overset{(r)}{\underline{\mathbf{K}}}_O \cdot \boldsymbol{\omega}\right)$$

The dimensionless $\underline{\mathbf{K}}$ dyadics are *intensive* properties of the body. They might aptly be called *specific* resistance dyadics, for they are independent of the *size* of the body—depending only upon its *shape*.

The direct dyadics are symmetric,

$$\overset{(t)}{\mathbf{K}} = \overset{(t)}{\mathbf{K}}{}^{\dagger} \tag{43a}$$

$$\overset{(r)}{\mathbf{K}}_O = \overset{(r)}{\mathbf{K}}_O{}^{\dagger} \tag{43b}$$

the latter relation holding at all origins O. Proof of these symmetry relations is based upon a *macroscopic* reciprocity theorem (B18, B22). This macroscopic proof is in marked contrast to the usual *microscopic* proof of the symmetry of the various transport coefficients via Onsager's reciprocal relations (cf. D4a). The symmetry of these coefficients indicates that no dissipationless (gyrostatic) forces or torques act on the particle (B22). In consequence of the nonnegative nature of the time rate of mechanical energy dissipation, these dyadics are necessarily positive-definite forms. The three eigenvalues of $\mathbf{K}$, say K_j ($j = 1, 2, 3$), corresponding to the three real roots of the cubic equation

$$\det\|\mathbf{K} - \mathbf{I}K\| = 0$$

are thus essentially positive for each of the direct dyadics. Furthermore, in consequence of the symmetry properties noted in Eq. (43), the normalized eigenvectors, $\mathbf{e}_j$ ($j = 1, 2, 3$), of $\mathbf{K}$, corresponding to the nontrivial solutions of the vector equation

$$\mathbf{K} \cdot \mathbf{e} = K\mathbf{e}$$

are thus mutually perpendicular. The direct dyadics may therefore be written in the canonical forms

$$\overset{(t)}{\mathbf{K}} = \overset{(t)}{\mathbf{e}}_1\overset{(t)}{\mathbf{e}}_1\overset{(t)}{K}_1 + \overset{(t)}{\mathbf{e}}_2\overset{(t)}{\mathbf{e}}_2\overset{(t)}{K}_2 + \overset{(t)}{\mathbf{e}}_3\overset{(t)}{\mathbf{e}}_3\overset{(t)}{K}_3 \tag{44}$$

and

$$\overset{(r)}{\mathbf{K}}_O = \overset{(r)}{\mathbf{e}}_1(O)\,\overset{(r)}{\mathbf{e}}_1(O)\,\overset{(r)}{K}_1(O) + \overset{(r)}{\mathbf{e}}_2(O)\,\overset{(r)}{\mathbf{e}}_2(O)\,\overset{(r)}{K}_2(O) + \overset{(r)}{\mathbf{e}}_3(O)\,\overset{(r)}{\mathbf{e}}_3(O)\,\overset{(r)}{K}_3(O) \tag{45}$$

where $|\mathbf{e}_j| = 1$, $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$, and $K_j > 0$, in which δ_{jk} is the Kronecker delta. The two triads of orthonormal vectors lie parallel to the principal axes of translation and to the principal axes of rotation at O, respectively.

Equations (38)–(39) are surprising in two respects. First, they show that the translational and rotational motions of a particle are generally coupled, in the sense that a rotary motion may give rise to a force, while a translatory motion may give rise to a torque. Second, what is equally surprising is the fact that the *same* coupling coefficient appears in both expressions. That this should be the case is by no means obvious; rather, demonstration of this fact requires use of the same macroscopic reciprocity principle (B18, B22) employed in the proof of Eqs. (43a, b). The identity of the cross coefficients in Eqs. (38)–(39), in conjunction with the symmetry relations (43), leads one to regard the former equations as Onsager relations. We emphasize, however, that this conclusion does not require the invocation of Onsager's reciprocity relations, but follows directly from conventional mechanical principles.

The fundamental resistance equations, (38) and (39), may be written as a single matrix equation (B22). Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be any triad of mutually perpendicular unit vectors, and consider the column matrices

$$\|\mathbf{F}\| = \begin{Vmatrix} F_1 \\ F_2 \\ F_3 \end{Vmatrix}, \qquad \|\mathbf{U}_O\| = \begin{Vmatrix} U_1(O) \\ U_2(O) \\ U_3(O) \end{Vmatrix}, \text{ etc.}$$

and the 3×3 square matrices

$$\left\|\overset{(r)}{\mathbf{K}}_O\right\| = \begin{Vmatrix} \overset{(r)}{K}_{11}(O) & \overset{(r)}{K}_{12}(O) & \overset{(r)}{K}_{13}(O) \\ \overset{(r)}{K}_{21}(O) & \overset{(r)}{K}_{22}(O) & \overset{(r)}{K}_{23}(O) \\ \overset{(r)}{K}_{31}(O) & \overset{(r)}{K}_{32}(O) & \overset{(r)}{K}_{33}(O) \end{Vmatrix}, \text{ etc.}$$

Further, define the 6×1 partitioned *wrench* matrix

$$\|\mathscr{F}_O\| = \begin{Vmatrix} \|\mathbf{F}\| \\ \|\mathbf{T}_O\| \end{Vmatrix} \tag{46}$$

the 6×1 partitioned *screw-velocity* matrix

$$\|\mathscr{U}_O\| = \left\| \begin{matrix} \|\mathbf{U}_O\| \\ \|\boldsymbol{\omega}\| \end{matrix} \right\| \tag{47}$$

and the 6×6 partitioned *resistance* matrix

$$\|\mathscr{K}_O\| = \left\| \begin{matrix} \|\overset{(t)}{\mathbf{K}}\| & \|\overset{(c)}{\mathbf{K}}_O\|^{\dagger} \\ \|\overset{(c)}{\mathbf{K}}_O\| & \|\overset{(r)}{\mathbf{K}}_O\| \end{matrix} \right\| \tag{48}$$

In terms of these we find

$$\|\mathscr{F}_O\| = -\mu \|\mathscr{K}_O\| \; \|\mathscr{U}_O\| \tag{49}$$

In a Stokes flow occurring within a fluid at rest at infinity, the rate of mechanical energy dissipation is equal to the rate at which the stresses acting over the particle surface are doing work; namely, $-\|\mathscr{U}_O\|^{\dagger}\|\mathscr{F}_O\|$. The dissipation rate in the entire fluid is thus $\mu\|\mathscr{U}_O\|^{\dagger}\|\mathscr{K}_O\| \; \|\mathscr{U}_O\|$. As the dissipation rate is essentially positive, and since the velocity $\|\mathscr{U}_O\|$ is arbitrary, this requires that the resistance matrix be positive-definite. Furthermore, in view of the symmetry of the translation and rotation submatrices, the resistance matrix is itself symmetric. According to a theorem of Frobenius (cf. M18) a symmetric (hermitian) matrix is positive-definite if and only if all the determinants of its principal minors are positive. This theorem ultimately gives rise to $2^6 - 1$ inequalities involving the scalar elements of the resistance matrix, though not all such inequalities are independent. The most important of these are (for $j \neq k$),

$$\begin{aligned}
&\overset{(t)}{K}_{jj} > 0, \qquad \overset{(t)}{K}_{jj}\overset{(t)}{K}_{kk} > \left(\overset{(t)}{K}_{jk}\right)^2, \quad \det\left\|\overset{(t)}{\mathbf{K}}\right\| > 0, \\
&\overset{(r)}{K}_{jj}(O) > 0, \qquad \overset{(r)}{K}_{jj}(O)\,\overset{(r)}{K}_{kk}(O) > \left(\overset{(r)}{K}_{jk}(O)\right)^2, \quad \det\left\|\overset{(r)}{\mathbf{K}}_O\right\| > 0, \\
&\overset{(t)}{K}_{jj}\overset{(r)}{K}_{kk}(O) > \left(\overset{(c)}{K}_{kj}(O)\right)^2, \qquad \det\|\mathscr{K}_O\| > 0
\end{aligned} \tag{50}$$

(no summation on the indices).

All the preceding equations are valid for any choice of origin. If the coupling and rotation dyadics are known at any point O they may be computed at any other point P by means of the *origin displacement theorems* (B22)

$$\overset{(c)}{\mathbf{K}}_P = \overset{(c)}{\mathbf{K}}_O - \mathbf{r}_{OP} \times \overset{(t)}{\mathbf{K}} \tag{51}$$

and

$$\overset{(r)}{\mathbf{K}}_P = \overset{(r)}{\mathbf{K}}_O - \mathbf{r}_{OP} \times \overset{(t)}{\mathbf{K}} \times \mathbf{r}_{OP} + \overset{(c)}{\mathbf{K}}_O \times \mathbf{r}_{OP} - \mathbf{r}_{OP} \times \overset{(c)}{\mathbf{K}}_O{}^{\dagger} \tag{52}$$

where $\mathbf{r}_{OP}$ is the vector drawn from O to P. The translation dyadic is independent of the choice of origin. These formulas derive from the fact that the hydrodynamic force on the particle must be independent of the choice of origin, whereas the torque will vary with choice of origin according to the relation $\mathbf{T}_P = \mathbf{T}_O - \mathbf{r}_{OP} \times \mathbf{F}$—the translational velocities at the two points being connected by the expression $\mathbf{U}_P = \mathbf{U}_O + \boldsymbol{\omega} \times \mathbf{r}_{OP}$.

For centrally symmetric bodies such as spheres, ellipsoids, and the like, it is intuitively obvious that the rotation and coupling dyadics will adopt their simplest and most symmetrical forms when expressed in terms of an origin at the center of symmetry. It is natural, therefore, to inquire as to the existence of a corresponding point for a body of arbitrary shape. Such an inquiry, based on Eqs. (51) and (52), discloses the fact (B22) that every body possesses a unique point, termed its *center of reaction* (R), at which the coupling dyadic is symmetric, i.e.,

$$\overset{(c)}{\mathbf{K}}_R = \overset{(c)}{\mathbf{K}}_R{}^{\dagger} \tag{53}$$

It is at this point that the dyadics assume their most physically significant forms. In general, this point does not coincide with the centroid of the body, though it does for centrally symmetric bodies. If the coupling dyadic is known at any point O the location of the center of reaction can be determined from the expression[5]

$$\mathbf{r}_{OR} = \left[\left(\mathbf{I} : \overset{(t)}{\mathbf{K}}\right)\mathbf{I} - \overset{(t)}{\mathbf{K}}\right]^{-1} \cdot \boldsymbol{\varepsilon} : \overset{(c)}{\mathbf{K}}_O \tag{54}$$

The symmetric dyadic $\overset{(c)}{\mathbf{K}}_R$ is identically zero for a large class of symmetrical particles, including spheres and ellipsoids. For such bodies Eqs. (38) and (39) adopt the simple forms

$$\mathbf{F} = -\mu \overset{(t)}{\mathbf{K}} \cdot \mathbf{U}_R \tag{55a}$$

$$\mathbf{T}_R = -\mu \overset{(r)}{\mathbf{K}}_R \cdot \boldsymbol{\omega} \tag{55b}$$

so that the translational and rotational motions become uncoupled. A nonzero value of $\overset{(c)}{\mathbf{K}}_R$ is therefore associated with propeller-like properties of the

[5] The superscript -1 denotes an inverse dyadic. With regard to multiple operations such as the indicated double-dot multiplications, we employ the "nesting convention" of Chapman and Cowling (C6). It is only in this one respect that we depart from the original convention of Gibbs (G3). According to the nesting convention the ordering of the operations is as follows: operations are performed from bottom to top; pairings of vectors are from the inside out; arrangement of products is from left to right. For example,

$$\mathbf{abc} \overset{\times}{\underset{\times}{\cdot}} \mathbf{defg} = (\mathbf{c} \times \mathbf{d})(\mathbf{b} \times \mathbf{e})(\mathbf{a} \cdot \mathbf{f})\, \mathbf{g} \ (= \text{a triad})$$

where $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{g}$ are any vectors.

body; that is, in the absence of external torques about R, a translational motion of the body generally causes the particle to spin—while, in the absence of external forces, a rotational motion of the particle generally causes it to translate. One should bear in mind that the source of such propeller-like behavior is not to be found in the inertia of the fluid, which ordinarily accounts for such action at large Reynolds numbers. Rather, the effect derives exclusively from the viscous stresses. It is important to note that only at the center of reaction does a nonzero value of the coupling dyadic imply propeller-like properties: for, in accordance with Eq. (51), even a sphere will give rise to a nonzero value of the coupling dyadic at any point other than its centroid. This merely reflects the fact that a sphere rotating about a noncentral axis will experience a net hydrodynamic force, while a translating sphere will experience a hydrodynamic torque about any point other than its geometric center (B18).

Two mirror-image (*enantiomorphic*) forms necessarily possess identical $\overset{(t)}{\mathbf{K}}$ and $\overset{(r)}{\mathbf{K}}_R$ values, and differ only in the algebraic sign of $\overset{(c)}{\mathbf{K}}_R$.

The values of the various resistance dyadics for a spherical particle of radius a are

$$\overset{(t)}{\mathbf{K}} = \mathbf{I}6\pi a, \qquad \overset{(r)}{\mathbf{K}}_R = \mathbf{I}8\pi a^3, \qquad \overset{(c)}{\mathbf{K}}_R = \mathbf{0} \tag{56}$$

R coincides with the sphere center. These dyadics are isotropic, as clearly they must be.

For the ellipsoidal particle

$$\frac{x_1^{\,2}}{a_1^{\,2}} + \frac{x_2^{\,2}}{a_2^{\,2}} + \frac{x_3^{\,2}}{a_3^{\,2}} = 1 \tag{57}$$

the three dyadics are (B23)

$$\overset{(t)}{\mathbf{K}} = 16\pi\left(\mathbf{e}_1\mathbf{e}_1 \frac{1}{\chi + a_1^{\,2}\alpha_1} + \mathbf{e}_2\mathbf{e}_2 \frac{1}{\chi + a_2^{\,2}\alpha_2} + \mathbf{e}_3\mathbf{e}_3 \frac{1}{\chi + a_3^{\,2}\alpha_3}\right) \tag{58}$$

$$\overset{(r)}{\mathbf{K}}_R = \frac{16\pi}{3}\left(\mathbf{e}_1\mathbf{e}_1 \frac{a_2^{\,2} + a_3^{\,2}}{a_2^{\,2}\alpha_2 + a_3^{\,2}\alpha_3} + \mathbf{e}_2\mathbf{e}_2 \frac{a_3^{\,2} + a_1^{\,2}}{a_3^{\,2}\alpha_3 + a_1^{\,2}\alpha_1} + \mathbf{e}_3\mathbf{e}_3 \frac{a_1^{\,2} + a_2^{\,2}}{a_1^{\,2}\alpha_1 + a_2^{\,2}\alpha_2}\right) \tag{59}$$

$$\overset{(c)}{\mathbf{K}}_R = \mathbf{0} \tag{60}$$

where R lies at the center of the ellipsoid and $\mathbf{e}_j$ is a unit vector parallel to Rx_j. In these expressions

$$\alpha_j = \int_0^\infty \frac{d\lambda}{(a_j^{\,2} + \lambda)\,\Delta(\lambda)} \qquad (j = 1, 2, 3) \tag{61}$$

$$\chi = \int_0^\infty \frac{d\lambda}{\Delta(\lambda)} \tag{62}$$

where

$$\Delta(\lambda) = [(a_1{}^2 + \lambda)(a_2{}^2 + \lambda)(a_3{}^2 + \lambda)]^{1/2} \tag{63}$$

For the slightly deformed sphere (A2, B21)

$$r = a\left[1 + \varepsilon \sum_{k=0}^{\infty} f_k(\theta, \phi)\right] \tag{64}$$

the three resistance dyadics are (B21)

$$\overset{(t)}{\mathbf{K}} = 6\pi a[\mathbf{I} + \varepsilon\{\mathbf{I}f_0 - \tfrac{1}{10}\nabla\nabla(r^2 f_2)\} + O(\varepsilon^2)] \tag{65}$$

$$\overset{(r)}{\mathbf{K}}_R = 8\pi a^3[\mathbf{I} + 3\varepsilon\{\mathbf{I}f_0 - \tfrac{1}{10}\nabla\nabla(r^2 f_2)\} + O(\varepsilon^2)] \tag{66}$$

$$\overset{(c)}{\mathbf{K}}_R = o(\varepsilon) \tag{67}$$

where ε is a small disposable constant; (r, θ, ϕ) are spherical coordinates having their origin either at the center O of the undeformed sphere or at the center of reaction of the deformed sphere; $f_k(\theta, \phi)$ is a surface spherical harmonic of degree k. The position of R relative to O is

$$\mathbf{r}_{OR} = \varepsilon a \nabla(r f_1) + O(\varepsilon^2) \tag{68}$$

To the first order in ε the center of reaction coincides with the centroid of the deformed sphere.

As an example of the application of these relations, consider the slightly deformed sphere obtained by setting

$$a_1 = a(1 + \varepsilon_1), \qquad a_2 = a(1 + \varepsilon_2), \qquad a_3 = a(1 + \varepsilon_3)$$

in Eq. (57), it being assumed that $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \ll 1$. Upon putting $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\phi$, and $x_3 = r\sin\theta\sin\phi$, and defining $\delta = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$, the equation of the ellipsoid (57) in polar form is found to be

$$r = a[1 + \varepsilon(f_0 + f_2)] + O(\varepsilon^2)$$

in which $\varepsilon f_0 = \delta P_0(\cos\theta) = \delta$ and

$$\varepsilon f_2 = (\varepsilon_1 - \delta)P_2(\cos\theta) + \tfrac{1}{6}(\varepsilon_2 - \varepsilon_3)P_2{}^2(\cos\theta)\cos 2\phi$$

The quantities P_n and $P_n{}^m$ are, respectively, Legendre and associated Legendre functions. It follows that

$$\varepsilon r^2 f_2 = (\varepsilon_1 - \delta)x_1{}^2 + (\varepsilon_2 - \delta)x_2{}^2 + (\varepsilon_3 - \delta)x_3{}^2$$

From Eqs. (65) and (66) the translation and rotation dyadics are found to be

$$\overset{(t)}{\mathbf{K}} = 6\pi a[\mathbf{I}(1 + \tfrac{6}{5}\delta) - \tfrac{1}{5}(\mathbf{e}_1\mathbf{e}_1\varepsilon_1 + \mathbf{e}_2\mathbf{e}_2\varepsilon_2 + \mathbf{e}_3\mathbf{e}_3\varepsilon_3)] + O(\varepsilon^2)$$

and

$$\overset{(r)}{\mathbf{K}}_R = 8\pi a^3[\mathbf{I}(1 + \tfrac{18}{5}\delta) - \tfrac{3}{5}(\mathbf{e}_1\mathbf{e}_1\varepsilon_1 + \mathbf{e}_2\mathbf{e}_2\varepsilon_2 + \mathbf{e}_3\mathbf{e}_3\varepsilon_3)] + O(\varepsilon^2)$$

The value $\delta = 0$ corresponds to the case where the volume of the ellipsoid is the same as that of a sphere of radius a, correctly to terms of $O(\varepsilon)$. These results may be confirmed by direct expansion of Eqs. (58) and (59) for small departures from the spherical shape.

From an operational point of view, equations of the form (65) and (66) are not entirely satisfactory, for they are not written in a truly *invariant* form. Given an irregular particle, where is the "axis" $|\cos\theta| = 1$, and how is the radius a of the "undeformed sphere" to be defined? The difficulty stems, of course, from the description of the surface of the particle in terms of the surface spherical harmonic expansion (64). From an operational viewpoint it is more consistent with the nature of the requisite physical measurements to express the size and shape of the body in terms of an expansion involving its volumetric moments. The kth such moment is

$$\int_V \overbrace{\mathbf{rr}\cdots\mathbf{rr}}^{k \text{ times}}\, dV$$

where $\mathbf{r}$ is measured from the centroid. Even more simply, one may resort to an expansion in terms of surface moments, the kth moment being

$$\int_S \overbrace{\mathbf{rr}\cdots\mathbf{rr}}^{k \text{ times}} \cdot d\mathbf{S}$$

In such terms, Eqs. (65) and (66) adopt the invariant forms (B21)

$$\overset{(t)}{\mathbf{K}} = 6\pi\left(\frac{V}{3S}\right)\left[\mathbf{I} + \left\{\frac{1}{10}\mathbf{I} - \frac{1}{8\pi}\left(\frac{3S}{V}\right)^5 \int_S d\mathbf{S}\cdot\mathbf{rrr}\right\} + O(\varepsilon^2)\right] \tag{69}$$

and

$$\overset{(r)}{\mathbf{K}}_R = 8\pi\left(\frac{V}{3S}\right)^3\left[\mathbf{I} + 3\left\{\frac{1}{10}\mathbf{I} - \frac{1}{8\pi}\left(\frac{3S}{V}\right)^5 \int_S d\mathbf{S}\cdot\mathbf{rrr}\right\} + O(\varepsilon^2)\right] \tag{70}$$

where V and S are, respectively, the volume of space occupied by the particle and its wetted surface area.

There are no other simple particle shapes for which the three resistance dyadics are wholly known, though some artificial bodies have been devised (B18, B22) to demonstrate that: (i) the center of reaction is not generally the same as the centroid; and (ii) bodies exist for which $\overset{(c)}{\mathbf{K}}_R \neq \mathbf{0}$.

Complete characterization of the hydrodynamic resistance of a solid particle in quasi-steady Stokes motion generally requires knowledge of 21 independent scalar resistance coefficients—six for the translation dyadic, six for the rotation dyadic, and nine for the coupling dyadic.[6] The number of such nonzero coefficients is reduced by virtue of any geometric symmetry that the particle may possess. Such a priori knowledge is useful in designing laboratory experiments to measure these phenomenological coefficients when they cannot be computed theoretically. The geometric symmetry investigation outlined by Brenner (B22) depends upon the fact that the direct resistance dyadics, regarded as tensors, transform under rotation and reflection as true (polar) second-rank tensors, whereas the coupling dyadic transforms as a second-rank pseudotensor[7] (axial tensor). In the following paragraphs some of the more significant results of the symmetry analysis are summarized.

If a body possesses three mutually perpendicular planes of reflection symmetry, its center of reaction lies at the point of intersection of these planes. The coupling dyadic is zero at this point, whereas the translation dyadic and rotation dyadic at R adopt the forms shown in Eqs. (44) and (45), in which the principal axes of translation and rotation (at R) coincide and lie normal to the three symmetry planes. An ellipsoid is an example of such a body [see Eqs. (58)–(60)].

If the shape of the particle is similarly related to each of the three mutually perpendicular planes, the resistance dyadics adopt the isotropic forms

$$\overset{(t)}{\mathbf{K}} = \mathbf{I}K_t \tag{71a}$$

$$\overset{(r)}{\mathbf{K}}_R = \mathbf{I}K_r \tag{71b}$$

$$\overset{(c)}{\mathbf{K}}_R = \mathbf{0} \tag{71c}$$

Such bodies are said to be *spherically isotropic*. Examples are spheres and the five regular polyhedrons—the tetrahedron, hexahedron (cube), octahedron,

[6] Only six coefficients are required to characterize the coupling dyadic at the center of reaction. But then an additional three scalars are required to specify the location of this point, so that the total number of independent scalars required for a complete characterization is still nine. Similarly, three scalars suffice for the translation dyadic if we refer them to the principal axes of translation [see Eq. (44)], but then three additional scalars (e.g., an appropriate set of Eulerian angles) are required to specify the orientations of these axes. So it comes down to the same thing—namely, that six scalars are required. The same is true of the rotation dyadic at any point, and of the coupling dyadic at the center of reaction.

[7] The components of a Cartesian pseudotensor satisfy the same transformation law as do those of a true Cartesian tensor of the same rank, except when the transformation alters the right- or left-handedness of the axes (an "improper" rotation), as for the case of reflection in a plane (*inversion*). In this case the two transformation laws differ by an algebraic sign.

dodecahedron, and icosahedron. It is remarkable that the latter have the same character as the sphere.

If the particle is a solid of revolution its center of reaction lies along the axis (say the Rx_1 axis). At this point the coupling dyadic vanishes, while both the translation and rotation dyadics at any point along the axis each adopt the general form

$$\mathbf{K} = \mathbf{I}K_{\parallel} + (\mathbf{I} - \mathbf{e}_1\mathbf{e}_1)K_{\perp} \tag{72}$$

where $\mathbf{e}_1$ is a unit vector along the symmetry axis, in either direction. The position of the center of reaction along the axis cannot be determined by inspection unless the body possesses fore-aft symmetry, i.e., a plane of reflection symmetry normal to the axis of revolution.

A particle possesses an axis of helicoidal symmetry if it is identical to itself when turned about that axis through the same angle ν in *either* direction (excluding the values $\nu = 0, \pi/2, \pi$). In such a case the center of reaction lies along this axis (say, the Rx_1 axis), while the translation, rotation, and coupling dyadics at R each have the form indicated in Eq. (72).

If a body possesses two distinct axes of helicoidal symmetry they must intersect. The point of intersection is then the center of reaction of the body. The three resistance dyadics are then isotropic at R:

$$\overset{(t)}{\mathbf{K}} = IK_t \tag{73a}$$

$$\overset{(r)}{\mathbf{K}}_R = \mathbf{I}K_r \tag{73b}$$

$$\overset{(c)}{\mathbf{K}}_R = \mathbf{I}K_c \tag{73c}$$

Such bodies are termed *helicoidally isotropic.* Their resistance is characterized by the two scalars K_t, K_r, and the pseudoscalar[8] K_c. Examples of isotropic helicoids are described by Brenner (B22). Spherical isotropy is clearly a special case of this more general type. On account of the inequalities set forth in Eq. (50), the various scalars satisfy the relations

$$K_t > 0 \tag{74a}$$

$$K_r > 0 \tag{74b}$$

$$K_t K_r > K_c^2 \tag{74c}$$

K_c is negative for "right-handed" (*dextral*) bodies. Conversely, it is positive

[8] A pseudoscalar is a pseudotensor of rank zero. This entity changes sign on reflection of the coordinates in a plane.

for "left-handed" (*sinistral*) bodies.[9] Enantiomorphic isotropic helicoids possess the same K_t and K_r values, differing only in the algebraic sign of K_c. The existence of such sense-dependent isotropic bodies is remarkable, though such behavior has long been known in another context (L7, pp. 337–343).

There exist a variety of three-dimensional bodies whose resistance dyadics are known only in part. Foremost among these are axially symmetric bodies for which the coefficients $\overset{(t)}{K}_{\|}$ and $\overset{(r)}{K}_{\|}$ appearing in Eq. (72) are known. In the translational case, formulas are available for toroids (P2, R4), lens-shaped bodies, including hemispheres and spherical caps (C16, P1), two-spheres (S20), and spindles (P3). Departing from the axisymmetric case, Roscoe (R6) and Gupta (G12) have developed electrostatic analogies which permit one to obtain the forces on flat disks of various cross sections undergoing translational motion perpendicular to the plane of the disk. Approximate methods have been proposed (B36, B37, T3, T8) for calculating the forces on finite circular cylinders. [See the pertinent discussion of Shi (S9a) bearing on the validity of such approximations.] Hill and Power (H13) have developed a pair of extremum principles which permit one to obtain upper and lower bounds for the drag on a body of arbitrary shape.

With regard to the rotation of axially symmetric bodies about their symmetry axes in the Stokes regime, explicit formulas for $\overset{(r)}{K}_{\|}$ are available for the following bodies: toroids (K1, R3), spherical caps (C15), lens-shaped bodies[10] (K1), two-spheres (J3, W1), and spindles (M3). A formula for $\overset{(r)}{K}_{\|}$ for a dumbbell consisting of two touching spheres, each of radius a, may be obtained as a limiting result from Jeffery's (J3) solution for the rotation of two equal spheres with equal angular velocities about their line of centers. This may be done by resorting to expansions of the type utilized in deriving Eq. (133a) from Eq. (130). The result ultimately obtained is

$$\overset{(r)}{K}_{\|} = 12\pi\zeta(3)a^3$$

where $\zeta(3) = 1.20206\ldots$ is the Riemann zeta function of argument 3. Since, for a single sphere, $\overset{(r)}{K}_{\|} = 8\pi a^3$, it follows from the fact that $12\pi\zeta(3)/(2 \times 8\pi) = 0.90154\ldots$ that the torque on the dumbbell is slightly *less* than on two

[9] A helicoidally isotropic body is said to be right-handed if, when it translates through the fluid, $\boldsymbol{\omega}$ is parallel to $\mathbf{U}_R$ (in the absence of an external torque about the center of reaction). Conversely, it is left-handed if $\boldsymbol{\omega}$ is antiparallel to $\mathbf{U}_R$ under these circumstances.

[10] Professor Kanwal has asked me to point out the following omission in his paper (K1): His Eq. (40) for the torque on a hemisphere should read (in our notation)

$$|T| = [8(135 - 59\sqrt{3})/81]\,\pi\mu a^3\omega + \tfrac{8}{3}\pi\mu a^3\omega \doteq 18.56\mu a^3\omega$$

where a is the radius.

spheres rotating separately. Kearsley (K7) gives a formula for obtaining an upper bound to the torque on a rotating body which holds for *all* angular Reynolds numbers, providing that the resulting motion is steady. This bound contains the Reynolds number as a parameter.

Collins (C14) points out a simple analogy for obtaining the Stokesian velocity field due to the rotation of an axisymmetric body about its axis, from a knowledge of the corresponding axially symmetric, streaming, *potential* flow of an incompressible ideal fluid past the body. When combined with Kanwal's (K1) formula, expressing the Stokes torque in terms of the asymptotic expansion of the velocity field at great distances from the rotating body, it yields a potent method for determining the torque on bodies whose surfaces are not members of any particular system of curvilinear coordinates. Thus, for example, airship forms and Rankine solids of a variety of shapes (M17) can be generated by an appropriate combination of point and line sources and sinks, doublets, etc., lying entirely within the body, providing that the total intake of the sinks equals the output of the sources. The combined Collins–Kanwal formula for the Stokes torque T about any point on the axis of the body rotating with angular speed ω in a fluid of viscosity μ is

$$|T| = 16\pi\mu\omega \lim_{r\to\infty} \frac{r^3\psi}{\varpi^2} \tag{75}$$

where ψ is the stream function for potential flow around the body with unit stream velocity, and r and ϖ are spherical and cylindrical coordinates, respectively, measured from any point on the axis. For example, for a prolate spheroid, Lamb (L5, p. 141) gives as the stream function for potential flow past this body

$$\psi = \tfrac{1}{2}\varpi^2 \frac{g(\zeta)}{g(\zeta_0)}$$

where

$$g(\zeta) = \frac{\zeta}{\zeta^2 - 1} - \tfrac{1}{2}\ln\frac{\zeta+1}{\zeta-1}$$

The spheroid dimensions are related to the spheroidal coordinate ζ_0 as follows:

$$a = c\zeta_0, \qquad b = c(\zeta_0{}^2 - 1)^{1/2}$$

where a and b are the polar and equatorial radii and $c = (a^2 - b^2)^{1/2}$ is half the distance between the focii of the spheroid. The couple is thus

$$|T| = \frac{8\pi\mu\omega}{g(\zeta_0)} \lim_{r\to\infty} r^3 g(\zeta)$$

As $\zeta \to \infty$, $r \to c\zeta$. Moreover

$$\ln\frac{\zeta+1}{\zeta-1} = \ln\frac{1+1/\zeta}{1-1/\zeta} = 2\left(\frac{1}{\zeta} + \frac{1}{3\zeta^3} + \frac{1}{5\zeta^5} + \cdots\right)$$

and

$$\frac{\zeta}{\zeta^2 - 1} = \frac{1}{\zeta}\left(\frac{1}{1 - 1/\zeta^2}\right) = \frac{1}{\zeta}\left(1 + \frac{1}{\zeta^2} + \frac{1}{\zeta^4} + \cdots\right)$$

An easy calculation then yields

$$\lim_{\zeta \to \infty} r^3 g(\zeta) = \tfrac{2}{3}c^3$$

whence

$$|T| = \frac{(16\pi/3)\mu\omega c^3}{(ac/b^2) - \frac{1}{2}\ln[(a + c)/(a - c)]}$$

in agreement with Jeffery's (J3) result for the prolate ellipsoid of revolution.

Accurate experimental data of recent vintage are available (C10, J1, J2a, H12, K11, M4, M11, P6) for the translational resistances of a large variety of simple, well-defined geometric shapes in the Stokes regime. (Earlier data are reviewed by these investigators.) These data are based on measurements of terminal settling velocities. Typical of the bodies studied were regular tetrahedrons, cubes, cube-octahedrons, rectangular prisms and finite cylinders of various aspect ratios, and square flat disks—to name a few. Many of these were studied for more than one orientation. These bodies, when homogeneous, are neutrally stable and can fall stably in any orientation, contrary to their usual behavior at higher Reynolds numbers. In most of these cases, sufficient experimental data are therefore available to permit evaluation of the complete translation dyadics. Jones and Knudsen (J6) measured the translational resistances of a number of bodies by towing them through a viscous oil, measuring, by means of a spring balance, the force required to impart a given velocity to the body. Their data show a large scatter owing partly to the inherent difficulties of the technique and partly to the question of how to correct for the resistance of the tow-wire itself.

Much of the data cited above originally included significant wall-effects, which the investigators attempted to correct for in various ways. Theoretically sound ways of making such corrections are discussed in Section II,C,3.

Experiments aimed at measuring the rotational resistances of simple bodies in infinite media appear nonexistent. Also lacking are experiments on screw- and propeller-like bodies (which rotate as they settle), as are quantitative experiments on the nonvertical fall of anisotropic bodies.[11]

[11] Experiments of this type are currently being pursued by N. Ganiaris (New York University) employing a nonvertically mounted column, inclined so as to be parallel to the direction of fall of the particle. In this way the particle always falls along the tube axis, never striking the wall.

Experimental data on the inclined fall of circular rods are presented by Mohr and Blumberg (M19). Their results, covering the complete range of possible orientations of the rod axis relative to the gravity field, agree with theoretical predictions.

Theoretical computation of the trajectory and ultimate state of motion of a solid particle of arbitrary shape falling under the influence of gravity in the Stokes regime is partially discussed by Brenner (B22). Such calculations involve a complex interplay between the three resistance dyadics, the moment of inertia dyadic, the relative positions of the centers of reaction, mass, and buoyancy, the mass and volume of the particle, the density and viscosity of the fluid, the gravity field vector, and the initial orientation, velocity, and spin of the particle. The difficulty implicit in such calculations stems from the fact that the nonscalar particle properties are constant only when referred to axes fixed in the particle, whereas the gravity vector is constant relative to axes fixed in space. As yet, no completely general theory of the "terminal" state of motion of a particle exists. Lacking are suitable criteria of the stability of such motions. However, in a few simple situations one is able to make definite assertions about the terminal motion (B22). Among the more interesting results are the following: (i) A homogeneous anisotropic particle such as an ellipsoid attains a neutrally stable terminal motion in which it settles without rotation. In general, it does not fall vertically but drifts horizontally as it settles, in a manner dependent upon the initial conditions of release; and (ii) A body such as an isotropic helicoid attains a neutrally stable terminal state in which it falls vertically and rotates about a vertical axis. The sense of the direction of rotation depends upon whether the helicoid is right or left handed.

The lack of suitable general stability criteria alluded to in the preceding paragraph has recently been partially overcome by Goldman, Cox, and Brenner (G5c).

Repeated sedimentation experiments performed on a randomly oriented, nonskew, neutrally stable, anisotropic particle (e.g., an ellipsoid) will, in general, display a spread in terminal settling velocities, owing to the dependence of settling velocity on orientation. Equivalently, a dilute suspension of identical, randomly oriented bodies will display a range of settling velocities. In these experiments the (gravity) force on each particle is the same. If all orientations are equally probable the average velocity $\overline{\mathbf{U}}$ will lie parallel to $\mathbf{F}$, so that the average translational resistance $\overline{K}_t$ in the formula $\overline{U} = -\mu^{-1}(\overline{K}_t)^{-1}\mathbf{F}$ is the *scalar* (B18)

$$(\overline{K}_t)^{-1} = \frac{1}{3}\,\text{trace}\left\| \overset{(t)}{\mathbf{K}} \right\|^{-1} = \frac{1}{3}\left(\overset{(t)}{K}{}_1^{-1} + \overset{(t)}{K}{}_2^{-1} + \overset{(t)}{K}{}_3^{-1}\right)$$

where $\overset{(t)}{K}{}_j^{-1} = 1/\overset{(t)}{K}_j$, in which the $\overset{(t)}{K}_j$ $(j = 1, 2, 3)$ are the eigenvalues of the translation dyadic. On the other hand, for streaming flow past a randomly oriented collection of identical particles, the stream velocity vector $\mathbf{U}$ will be the same for each particle and the average force $\overline{\mathbf{F}}$ will lie parallel to $\mathbf{U}$. The average resistance $\bar{K}_t$ in the formula $\bar{\mathbf{F}} = \mu \bar{K}_t \mathbf{U}$ is therefore, once again, a scalar, but it is now given by the formula

$$\bar{K}_t = \frac{1}{3} \operatorname{trace} \left\| \overset{(t)}{\mathbf{K}} \right\| = \frac{1}{3} \left(\overset{(t)}{K_1} + \overset{(t)}{K_2} + \overset{(t)}{K_3} \right)$$

These two average resistances are not the same, though the difference is not likely to be large. [For example, for a circular disk of radius a ($\overset{(t)}{K_1} = \overset{(t)}{K_2} = \frac{32}{3}a$, $K_3 = 16a$) the two average resistances are $12a$ and $12\frac{4}{9}a$, respectively.] Nevertheless, the calculation does reveal the need for caution when utilizing *average* results in the form $\mathbf{F} = -\mu \bar{K}_t \mathbf{U}$.

In a somewhat similar vein, a clear distinction must be made between the true translational resistance dyadic $\overset{(t)}{\mathbf{K}}$ of a skewed particle and the *apparent* value thereof that would be directly observed in a settling velocity experiment. If, in Eq. (39), O is the point about which the net gravitational plus buoyant torques on a freely settling particle vanish, we find upon setting $\mathbf{T}_O = \mathbf{0}$ and eliminating $\boldsymbol{\omega}$ between Eqs. (38) and (39) that

$$\mathbf{F} = -\mu \left[\overset{(t)}{\mathbf{K}} - \overset{(c)}{\mathbf{K}_O}{}^{\dagger} \cdot \overset{(r)}{\mathbf{K}_O}{}^{-1} \cdot \overset{(c)}{\mathbf{K}_O} \right] \cdot \mathbf{U}_O$$

The dyadic coefficient of proportionality between the force and translational velocity in this expression is, therefore, not the same as the true translational resistance dyadic in the formula $\mathbf{F} = \mu \overset{(t)}{\mathbf{K}} \cdot \mathbf{U}$, applicable to the case of, say, streaming flow past the *immobilized* particle (with stream velocity $\mathbf{U}$). A homogeneous, isotropic helicoid furnishes a simple, but definite example of this distinction; for if O refers to its center, the outcome of the settling velocity experiment corresponds to the formula [see Eq. (71)]

$$\mathbf{F} = -\mu \left[K_t - \frac{K_c^2}{K_r} \right] \mathbf{U}_O$$

In accordance with Eq. (74) the apparent translational resistance of this particle is positive-definite.

2. *Shear and Higher-Order Flows*

a. Solid Particles. In this section we discuss the quasi-static Stokes resistance of a solid particle of any shape immersed in a fluid which is in a state of net motion at infinity (B23, B24, B26). The fluid motion is thus governed by the quasi-steady Stokes equations, the continuity equation for incompressible fluids, and the boundary conditions

$$\mathbf{v} = \mathbf{U} + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{on } B \tag{76}$$

$$\mathbf{v} \to \mathbf{u} \qquad \text{at infinity} \tag{77}$$

where $\mathbf{U}$ is the velocity of an arbitrary point O affixed to the particle and $\mathbf{r}$ is measured from O. The arbitrary field $\mathbf{u} = \mathbf{u}(\mathbf{r})$ at infinity must itself satisfy the equations of motion—otherwise the system of equations and boundary conditions are incompatible. Examples of such compatible flows at infinity are shear and Poiseuille (parabolic) flows, for these satisfy Stokes equations identically.

Rather remarkably, the forces and torques experienced by particles in such arbitrary flows can be obtained solely from knowledge of the corresponding translational and rotational solutions of Stokes equations *for the case where the fluid is at rest at infinity.* Explicitly, if $\overset{(t)}{\mathbf{\Pi}}$ and $\overset{(r)}{\mathbf{\Pi}}_O$ are the intrinsic translational and rotational triadic stress fields for the particle in question, defined in Eqs. (35) and (36), the force and torque experienced by the body in an arbitrary flow are (B24, B26)

$$\mathbf{F} = \mu \int_B d\mathbf{S} \cdot \overset{(t)}{\mathbf{\Pi}}{}^{\dagger} \cdot (\mathbf{v} - \mathbf{u}) \tag{78}$$

$$\mathbf{T}_O = \mu \int_B d\mathbf{S} \cdot \overset{(r)}{\mathbf{\Pi}}_O{}^{\dagger} \cdot (\mathbf{v} - \mathbf{u}) \tag{79}$$

where $\mathbf{u}(\mathbf{r})$ is the arbitrarily prescribed Stokes flow field at infinity and $\mathbf{v}$ refers to the velocity field at the surface B of the particle. These relations do not actually require that the particle be solid, but rather apply for any arbitrary field $\mathbf{v}$ at the particle surface. By expressing the intrinsic stress triadics in trinomial form in an arbitrary system of Cartesian coordinates (x_1, x_2, x_3),

$$\overset{(\,)}{\mathbf{\Pi}} = \frac{1}{\mu}\left[\overset{(\,)}{\boldsymbol{\pi}}_1\mathbf{i}_1 + \overset{(\,)}{\boldsymbol{\pi}}_2\mathbf{i}_2 + \overset{(\,)}{\boldsymbol{\pi}}_3\mathbf{i}_3\right]$$

Eqs. (78) and (79) may be written alternatively in terms of their Cartesian components,

$$F_j = \int_B d\mathbf{S}\cdot\overset{(t)}{\boldsymbol{\pi}}{}^{[j]}\cdot(\mathbf{v} - \mathbf{u}) \qquad (j = 1, 2, 3)$$

$$(T_O)_j = \int_B d\mathbf{S}\cdot\overset{(r)}{\boldsymbol{\pi}}_O{}^{[j]}\cdot(\mathbf{v} - \mathbf{u}) \qquad (j = 1, 2, 3)$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are unit vectors parallel to the coordinate axes. Here, $\overset{(t)}{\boldsymbol{\pi}}{}^{[j]}$ is the conventional stress dyadic [cf. Eq. (3)] for translational motion of the particle with *unit* velocity in the x_j direction in a fluid at rest at infinity, and $\overset{(r)}{\boldsymbol{\pi}}_O{}^{[j]}$ is the stress dyadic for rotation of the particle about an axis through O with *unit* angular velocity, parallel to the x_j axis, in a fluid at rest at infinity. In these expressions $F_j = \mathbf{i}_j\cdot\mathbf{F}$, with a similar expression for the torque.

If the particle is solid, $\mathbf{v}$ is given on B by Eq. (76). It is convenient to expand $\mathbf{u}$ in a *symbolic* Taylor series[12]

$$\mathbf{u}(\mathbf{r}) = \exp(\mathbf{r} \cdot \nabla_O)\, \mathbf{u}(\mathbf{r}_O) \tag{80}$$

where O, once again, refers to the arbitrary point moving with the particle. We then find from Eqs. (78) and (79) that (B26)

$$\mathbf{F} = -\mu \mathfrak{F} \cdot (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r} - \mathbf{u}) \tag{81}$$

$$\mathbf{T}_O = -\mu \mathfrak{T}_O \cdot (\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r} - \mathbf{u}) \tag{82}$$

where the dyadic entities

$$\mathfrak{F} = -\int_B d\mathbf{S} \cdot \overset{(t)}{\boldsymbol{\Pi}}{}^{\dagger} \exp(\mathbf{r} \cdot \nabla_O) \tag{83}$$

and

$$\mathfrak{T}_O = -\int_B d\mathbf{S} \cdot \overset{(r)}{\boldsymbol{\Pi}}{}_O^{\dagger} \exp(\mathbf{r} \cdot \nabla_O) \tag{84}$$

are *symbolic operators*. The nabla operator ∇_O is to be regarded as a constant in the $\mathbf{r}$-space integration. These intrinsic force and torque operators depend only on the exterior shape and size of the particle, though the latter operator also depends upon the location of O. These dyadic differential operators operate on the "slip velocity" between particle and fluid, $\mathbf{U} + \boldsymbol{\omega} \times \mathbf{r} - \mathbf{u}$. In a broader sense, Eqs. (81) and (82) express the fact that the force and torque are *linear vector functionals* of the difference between the particle and fluid velocities. An alternative form of these relations is

$$\mathbf{F} = -\mu\left(\overset{(t)}{\mathbf{K}} \cdot \mathbf{U}_O + \overset{(c)}{\mathbf{K}}{}_O^{\dagger} \cdot \boldsymbol{\omega} - \mathfrak{F} \cdot \mathbf{u}\right) \tag{85}$$

$$\mathbf{T}_O = -\mu\left(\overset{(c)}{\mathbf{K}}{}_O \cdot \mathbf{U}_O + \overset{(r)}{\mathbf{K}}{}_O \cdot \boldsymbol{\omega} - \mathfrak{T}_O \cdot \mathbf{u}\right) \tag{86}$$

[12] That is,

$$\mathbf{u}(x, y, z) = \left[\exp\left(x \frac{\partial}{\partial x_O} + y \frac{\partial}{\partial y_O} + z \frac{\partial}{\partial z_O}\right)\right] \mathbf{u}(x_O, y_O, z_O)$$

Symbolic methods are useful not only for representing resistance formulas but also for obtaining detailed solutions of the equations of motion for arbitrary boundary conditions. The general methods have much in common with the use of Green's functions (F12, L5a, M20) to represent the solution for arbitrary boundary conditions. However, in contrast with Green's functions, which express the solution in the form of an *integral operator* operating on the boundary conditions, the solution appears here in the form of a symbolic *differential operator* operating on the boundary conditions. The connection between the two methods lies in the fact that when the boundary data are sufficiently smooth the Green's function integrals can be evaluated by resorting to symbolic expansions of the type given by Eq. (80).

Since the translational and rotational Stokes problems are completely solved for the sphere and the ellipsoid, one can evaluate these symbolic operators for such bodies. For the sphere (B26)

$$\mathfrak{F} = \mathbf{I}6\pi a \frac{\sinh(a\nabla_R)}{a\nabla_R} \tag{87}$$

and

$$\mathfrak{T}_R = -\frac{12\pi a^3}{(a\nabla_R)^2}\left[\cosh(a\nabla_R) - \frac{\sinh(a\nabla_R)}{a\nabla_R}\right] \boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla}_R \tag{88}$$

where R is the sphere center. Here, $\nabla = (\nabla^2)^{1/2}$, where ∇^2 is the ordinary Laplace operator. Note carefully the distinction between the scalar operator ∇ and the ordinary vector nabla operator $\boldsymbol{\nabla}$. The symbolic operators appearing above may be interpreted by employing the infinite series expansions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

Substitution into (81) and (82) eventually yields

$$\mathbf{F} = -6\pi\mu a\left[(\mathbf{U}_R - \mathbf{u}_R) - \frac{a^2}{6}(\nabla^2\mathbf{u})_R\right] \tag{89a}$$

$$\mathbf{T}_R = -8\pi\mu a^3[\boldsymbol{\omega} - \tfrac{1}{2}(\boldsymbol{\nabla} \times \mathbf{u})_R] \tag{89b}$$

where the affix R implies evaluation at the sphere center. In obtaining these it has been noted that $(\boldsymbol{\varepsilon}\cdot\boldsymbol{\nabla})\cdot\mathbf{u} = -\boldsymbol{\nabla} \times \mathbf{u}$. We have also utilized the fact that since $\mathbf{u}$ satisfies Stokes equations, then $\nabla^4\mathbf{u} = \mathbf{0}$ and $\nabla^2(\boldsymbol{\nabla} \times \mathbf{u}) = \mathbf{0}$, whereupon all the higher-order terms in the original infinite series expansions vanish. Equations (89) are Faxén's laws (F2, P4). They directly furnish the force and torque on a solid spherical particle translating and spinning within a fluid in an arbitrary state of Stokes flow at infinity. The last term in Eq. (89a) may be expressed in alternative form by noting that $\nabla^2\mathbf{u} = \mu^{-1}\boldsymbol{\nabla}p^0$, where p^0 is the unperturbed pressure field.

The symbolic operators (87)–(88) for the sphere possess a greater degree of generality than do Faxén's laws. In particular, if we consider any Stokes flow $\mathbf{v}(r, \mathbf{r}/r)$ vanishing at infinity and satisfying the arbitrary boundary condition $\mathbf{v} = \mathbf{f}(\mathbf{r}/r) \equiv \mathbf{f}(\theta, \phi)$ at $r = a$, then the force on the sphere may be obtained directly from the prescribed velocity boundary condition via the expression

$$\mathbf{F} = -6\pi\mu a \frac{\sinh(a\nabla_R)}{a\nabla_R}\mathbf{f}\left(\frac{\mathbf{r}_R}{a}\right)$$

or, what is equivalent,

$$\mathbf{F} = -6\pi\mu a\left[\frac{\sinh \nabla}{\nabla}\mathbf{f}(\mathbf{r})\right]_R \tag{90a}$$

The function $\mathbf{f}(\mathbf{r})$ need not satisfy Stokes equations, nor indeed any conditions other than that of analyticity at the origin. Similarly, for the torque we have

$$\mathbf{T}_R = -12\pi\mu a^2\left[\frac{1}{\nabla^2}\left(\cosh \nabla - \frac{\sinh \nabla}{\nabla}\right)\nabla \times \mathbf{f}(\mathbf{r})\right]_R \tag{90b}$$

Suppose, for example, that the prescribed boundary condition is

$$\mathbf{v} = \mathbf{i}_1(A + B\sin^2\theta + C\sin\theta\sin\phi) \qquad \text{at } r = a$$

where A, B, C are constants and (r, θ, ϕ) are spherical coordinates related to the Cartesian system (x_1, x_2, x_3) as in the example following Eq. (68). This makes

$$\mathbf{f}(\mathbf{r}) = \mathbf{i}_1[A + B(x_2{}^2 + x_3{}^2) + Cx_3]$$

which ultimately yields

$$\mathbf{F} = -\mathbf{i}_1 6\pi\mu a(A + \tfrac{2}{3}B)$$

$$\mathbf{T}_R = -\mathbf{i}_2 4\pi\mu a^2 C$$

in agreement with results cited by Brenner and Happel (B28). Formulas of the type (90) are inapplicable unless $\mathbf{f}(\mathbf{r})$ is infinitely differentiable at the origin, in which case one must revert to the original integral formulas (78)–(79). For the sphere these reduce to (B24)

$$\mathbf{F} = -\tfrac{3}{2}\mu a \int_{S_1} \mathbf{f}\left(\frac{\mathbf{r}}{r}\right) d\Omega \tag{91a}$$

$$\mathbf{T}_R = -3\mu a^2 \int_{S_1} \frac{\mathbf{r}}{r} \times \mathbf{f}\left(\frac{\mathbf{r}}{r}\right) d\Omega \tag{91b}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is an element of surface area on a unit sphere, S_1. The relationship between the integral forms (91) and the differential forms (90) is readily grasped by considering the equivalent one-dimensional relation

$$\int_{-a}^{a} f(x)\,dx = 2\,\frac{\sinh\{a(d/dx_O)\}}{(d/dx_O)} f(x_O)$$

for any $f(x)$. That symbolic methods have attained no systematic degree of popularity appears to be due in large measure to the preoccupation of mathematicians with pathological functions for which such integration formulas are invalid. Where applicable, such methods are very potent, for they replace integration by differentiation.

The symbolic operators for the ellipsoid, Eq. (57), are (B26)

$$\mathfrak{F} = \overset{(t)}{\mathbf{K}} \frac{\sinh D_R}{D_R} \tag{92a}$$

and

$$\mathfrak{T}_R = -3\left\{\frac{1}{D_R}\frac{d}{dD_R}\left(\frac{\sinh D_R}{D_R}\right)\right\}\mathbf{Q}\cdot\boldsymbol{\varepsilon}\cdot\Box_R \tag{92b}$$

where R refers to the center of the ellipsoid, i.e., its center of reaction; $\overset{(t)}{\mathbf{K}}$ is the translation dyadic for the ellipsoid, given in Eq. (58); $D = (D^2)^{1/2}$ is a scalar operator in which

$$D^2 = a_1{}^2 \frac{\partial^2}{\partial x_1{}^2} + a_2{}^2 \frac{\partial^2}{\partial x_2{}^2} + a_3{}^2 \frac{\partial^2}{\partial x_3{}^2} \tag{93}$$

is a sort of distorted Laplace operator; $\mathbf{Q}$ is the constant dyadic

$$\mathbf{Q} = \frac{16\pi}{3}\left[\frac{\mathbf{e}_1\mathbf{e}_1}{a_2{}^2\alpha_2 + a_3{}^2\alpha_3} + \frac{\mathbf{e}_2\mathbf{e}_2}{a_3{}^2\alpha_3 + a_1{}^2\alpha_1} + \frac{\mathbf{e}_3\mathbf{e}_3}{a_1{}^2\alpha_1 + a_2{}^2\alpha_2}\right] \tag{94}$$

in which $\mathbf{e}_j$ ($j = 1, 2, 3$) is a unit vector drawn parallel to a principal axis Rx_j of the ellipsoid, and α_j is defined in Eq. (61). Lastly, $\Box$ is the vector differential operator

$$\Box = \mathbf{e}_1 a_1{}^2 \frac{\partial}{\partial x_1} + \mathbf{e}_2 a_2{}^2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 a_3{}^2 \frac{\partial}{\partial x_3} \tag{95}$$

Substitution into Eqs. (81)–(82) or (85)–(86) yields

$$\mathbf{F} = -\mu\overset{(t)}{\mathbf{K}}\cdot\left[\mathbf{U} - \left(\frac{\sinh D}{D}\right)\mathbf{u}\right]_R \tag{96}$$

and

$$\mathbf{T}_R = -\mu\overset{(r)}{\mathbf{K}}_R\cdot\left[\boldsymbol{\omega} - 3\mathbf{A}\cdot\left\{\frac{1}{D}\frac{d}{dD}\left(\frac{\sinh D}{D}\right)\right\}\Box\times\mathbf{u}\right]_R \tag{97}$$

$\overset{(r)}{\mathbf{K}}_R$ is given in Eq. (59); $\mathbf{A}$ is the constant dyadic

$$\mathbf{A} = \frac{\mathbf{e}_1\mathbf{e}_1}{a_2{}^2 + a_3{}^2} + \frac{\mathbf{e}_2\mathbf{e}_2}{a_3{}^2 + a_1{}^2} + \frac{\mathbf{e}_3\mathbf{e}_3}{a_1{}^2 + a_2{}^2}$$

When the operators appearing in (96) and (97) are expanded via their infinite series representations the resulting expansions do not terminate after a finite number of terms, as do Faxén's laws for the sphere, Eqs. (89).

To illustrate the applicability of these relations consider the case of an ellipsoid in a uniform shear field [see Eq. (106), where O is taken at the center of the ellipsoid]. From Eqs. (81)–(82), (92), and (106) one finds, upon expanding the hyperbolic functions, that

$$\mathbf{F} = -\mu \overset{(t)}{\mathbf{K}} \cdot \left[\left(1 + \frac{D^2}{6} + \frac{D^4}{120} + \cdots\right)\{\mathbf{U} - \mathbf{u} + (\boldsymbol{\omega} - \boldsymbol{\omega}_f) \times \mathbf{r} - \mathbf{S} \cdot \mathbf{r}\}\right]_R$$

and

$$\mathbf{T}_R = -\mu \mathbf{Q} \cdot \left[\left(1 + \frac{D^2}{10} + \frac{D^4}{280} + \cdots\right)\square \times \{(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \times \mathbf{r} - \mathbf{S} \cdot \mathbf{r}\}\right]_R$$

where it has been noted that $\square \times (\mathbf{U}_R - \mathbf{u}_R) = \mathbf{0}$, as follows from the constancy of the vectors $\mathbf{U}_R$ and $\mathbf{u}_R$. Since $D^2\mathbf{r} = \mathbf{0}$ everywhere, and since $\mathbf{r} = \mathbf{0}$ at the center of the ellipsoid, these reduce to

$$\mathbf{F} = -\mu \overset{(t)}{\mathbf{K}} \cdot (\mathbf{U}_R - \mathbf{u}_R)$$

and

$$\mathbf{T}_R = -\mu \mathbf{Q} \cdot [\square \times \{(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \times \mathbf{r} - \mathbf{S} \cdot \mathbf{r}\}]$$

If the sphere is neutrally buoyant the former relation requires that $\mathbf{U}_R = \mathbf{u}_R$, so that the center of the ellipsoid is carried along with the fluid. We note the identities $\square \times \{(\boldsymbol{\omega} - \boldsymbol{\omega}_f) \times \mathbf{r}\} = (\boldsymbol{\omega} - \boldsymbol{\omega}_f) \cdot (\mathbf{I}\square \cdot \mathbf{r} - \square\mathbf{r})$ and $\square \times (\mathbf{S} \cdot \mathbf{r}) = -\mathbf{S}^\dagger \times \square \cdot \mathbf{r}$. Thus, in terms of a system of Cartesian coordinates instantaneously coinciding with the principal axes of the ellipsoid, the following expressions are obtained for the components of the couple about the center of the ellipsoid:

$$T_1 = \frac{16\pi\mu}{3(a_2{}^2\alpha_2 + a_3{}^2\alpha_3)}\{(a_2{}^2 - a_3{}^2)S_{23} + (a_2{}^2 + a_3{}^2)(\omega_1{}^f - \omega_1)\}$$

$$T_2 = \frac{16\pi\mu}{3(a_3{}^2\alpha_3 + a_1{}^2\alpha_1)}\{(a_3{}^2 - a_1{}^2)S_{31} + (a_3{}^2 + a_1{}^2)(\omega_2{}^f - \omega_2)\}$$

$$T_3 = \frac{16\pi\mu}{3(a_1{}^2\alpha_1 + a_2{}^2\alpha_2)}\{(a_1{}^2 - a_2{}^2)S_{12} + (a_1{}^2 + a_2{}^2)(\omega_3{}^f - \omega_3)\}$$

in which $S_{ij} = \mathbf{e}_j\mathbf{e}_i : \mathbf{S}$. The validity of the preceding relations depends upon the symmetry of $\mathbf{S}$. These results agree identically with those originally given by Jeffery (J4). Jeffery, however, obtained his formulas by actually solving the boundary-value problem appropriate to an ellipsoid at incidence in a shear field.

For the slightly deformed sphere described by Eq. (64) Ripps and Brenner (R5a) give the following expression for the symbolic dyadic force operator:

$$\mathfrak{F} = 6\pi a\left[\mathbf{I}\frac{\sinh(a\nabla_R)}{a\nabla_R} + \varepsilon\sum_{n=0}^{\infty}\left\{\left(2n + 1 + a\nabla_R\frac{d}{d(a\nabla_R)}\right)\mathbf{I}\mathfrak{L}_n^{(0)}(a\nabla_R)\right.\right.$$
$$\left.\left. - \frac{(2n+1)}{(n+2)(2n+5)}\mathfrak{L}_n^{(2)}(a\nabla_R)\right\}\left\{\frac{1}{a\nabla_R}\frac{d}{d(a\nabla_R)}\right\}^n\frac{\sinh(a\nabla_R)}{a\nabla_R}\right] + O(\varepsilon^2) \quad (98)$$

in which

$$\mathfrak{L}_n^{(0)}(a\nabla_R) = \frac{1}{n!}\,\mathbf{C}_n \,\boxed{n}\, (a\nabla_R)^n$$

and

$$\mathfrak{L}_n^{(2)}(a\nabla_R) = \frac{1}{n!}\,\mathbf{C}_{n+2} \,\boxed{n}\, (a\nabla_R)^n$$

are, respectively, scalar and dyadic operators of argument $a\nabla_R$. Here

$$\mathbf{C}_k = \overbrace{\nabla\nabla \cdots \nabla\nabla}^{k \text{ times}} (r^k f_k)$$

is a (dimensionless) *constant* k-adic and

$$(a\nabla_R)^n = a^n \overbrace{\nabla_R\nabla_R \cdots \nabla_R\nabla_R}^{n \text{ times}}$$

is a (dimensionless) n-adic differential operator in "R-space." The symbol $\boxed{n}$ denotes n successive polydot multiplications. The two $\mathfrak{L}$-operators may also be written, respectively, in the following scalar and dyadic forms:

$$\mathfrak{L}_n^{(0)}(a\nabla_R) = \frac{a^n}{n!}(\nabla \cdot \nabla_R)^n (r^n f_n)$$

$$\mathfrak{L}_n^{(2)}(a\nabla_R) = \frac{a^n}{n!}\nabla\nabla(\nabla \cdot \nabla_R)^n (r^{n+2} f_{n+2})$$

where

$$\nabla \cdot \nabla_R \equiv \frac{\partial}{\partial x}\frac{\partial}{\partial x_R} + \frac{\partial}{\partial y}\frac{\partial}{\partial y_R} + \frac{\partial}{\partial z}\frac{\partial}{\partial z_R}$$

bearing in mind that the R-space differentiations do not operate on the physical space variables (r, θ, ϕ) or (x, y, z). The force operator may be interpreted via the expansion

$$\left(\frac{1}{\xi}\frac{d}{d\xi}\right)^n \frac{\sinh \xi}{\xi} = \frac{1}{\xi^n}\left(\frac{\pi}{2\xi}\right)^{1/2} I_{n+1/2}(\xi)$$

$$= \frac{2^n n!}{(2n+1)!}\left[1 + \frac{\xi^2}{2(2n+3)} + \frac{\xi^4}{2 \cdot 4(2n+3)(2n+5)} + \cdots\right]$$

where $I_{n+1/2}$ is the modified Bessel function of the first kind of order $n + \frac{1}{2}$. The comparable expression for the symbolic dyadic torque operator for the slightly deformed sphere is also available (R5a).

A systematic investigation of the number and nature of the nonzero coefficients of the symbolic force dyadic and torque pseudodyadic operators may be made for bodies possessing various types of geometric symmetry.

In certain applications it is physically more enlightening to express the operator equations (81) and (82) in terms of polyadic resistance coefficients (B23, B24), rather than leaving them in their present symbolic forms. This may be achieved by utilizing the expansion

$$\begin{aligned}\exp(\mathbf{r}\cdot\nabla_O) &= 1 + \frac{1}{1!}\mathbf{r}\cdot\nabla_O + \frac{1}{2!}\mathbf{rr}:\nabla_O\nabla_O + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{r}^n \boxed{n}\ \nabla_O{}^n\end{aligned} \tag{99}$$

where $\mathbf{r}^n$ is the n-adic

$$\mathbf{r}^n = \overbrace{\mathbf{rr}\cdots\mathbf{rr}}^{n\text{ times}}$$

in which $\mathbf{r}$ is measured from O; $\nabla_O{}^n$ is the n-adic differential operator

$$\nabla_O{}^n = \overbrace{\nabla_O\nabla_O\cdots\nabla_O\nabla_O}^{n\text{ times}}$$

$\boxed{n}$ denotes n successive polydot multiplications. In this notation the symbolic force and torque operators defined in (83) and (84) may be written as

$$\mathfrak{F} = \sum_{n=0}^{\infty} {}_{n+2}\overset{(F)}{\mathfrak{K}}_O \boxed{n}\ \nabla_O{}^n \tag{100}$$

and

$$\mathfrak{T}_O = \sum_{n=0}^{\infty} {}_{n+2}\overset{(T)}{\mathfrak{K}}_O \boxed{n}\ \nabla_O{}^n \tag{101}$$

in which

$${}_{n+2}\overset{(F)}{\mathfrak{K}}_O = -\frac{1}{n!}\int_B d\mathbf{S}\cdot\overset{(t)}{\mathbf{\Pi}}{}^{\dagger}\mathbf{r}^n \tag{102}$$

is a constant $(n+2)$-adic intrinsic force resistance coefficient, and

$${}_{n+2}\overset{(T)}{\mathfrak{K}}_O = -\frac{1}{n!}\int_B d\mathbf{S}\cdot\overset{(r)}{\mathbf{\Pi}}_O{}^{\dagger}\mathbf{r}^n \tag{103}$$

is a constant, pseudo$(n+2)$-adic intrinsic torque resistance coefficient (B24). Thus, from (100) and (101)

$$\mathfrak{F}\cdot\mathbf{u} = \sum_{n=0}^{\infty} {}_{n+2}\overset{(F)}{\mathfrak{K}}_O \boxed{n+1}\ (\nabla^n\mathbf{u})_O \tag{104}$$

$$\mathfrak{T}_O\cdot\mathbf{u} = \sum_{n=0}^{\infty} {}_{n+2}\overset{(T)}{\mathfrak{K}}_O \boxed{n+1}\ (\nabla^n\mathbf{u})_O \tag{105}$$

Substitution into (85) and (86) yields expressions for the force and torque in terms of these polyadic resistance coefficients.

For a sphere of radius a these polyadic resistance coefficients are, upon choosing the origin O at the sphere center (B24),

$$
{}_{n+2}\overset{(F)}{\mathfrak{K}}_O = \begin{cases} 6\pi a\mathbf{I} & \text{for } n = 0 \\ \pi a^3\mathbf{II} & \text{for } n = 2 \\ \mathbf{0} & \text{for all other } n \end{cases}
$$

$$
{}_{n+2}\overset{(T)}{\mathfrak{K}}_O = \begin{cases} -4\pi a^3\boldsymbol{\varepsilon} & \text{for } n = 1 \\ \mathbf{0} & \text{for all other } n \end{cases}
$$

The corresponding results for the ellipsoid, Eq. (57), relative to an origin at its center, are (B24)

$$
{}_{n+2}\overset{(F)}{\mathfrak{K}}_O = \begin{cases} \dfrac{1}{n!(n+1)!}\overset{(t)}{\mathbf{K}}(D^2)^{n/2}\mathbf{r}^n & \text{for } n \text{ even} \\ \mathbf{0} & \text{for } n \text{ odd} \end{cases}
$$

$$
{}_{n+2}\overset{(T)}{\mathfrak{K}}_O = \begin{cases} -\dfrac{3}{n!(n+2)!}\mathbf{Q}\cdot\boldsymbol{\varepsilon}\cdot(D^2)^{(n+1)/2}\mathbf{r}^{n+1} & \text{for } n \text{ odd} \\ \mathbf{0} & \text{for } n \text{ even} \end{cases}
$$

where the constant dyadics $\overset{(t)}{\mathbf{K}}$ and $\mathbf{Q}$ are defined in Eqs. (58) and (94), respectively. The differential operator D^2 is defined in Eq. (93), and $\mathbf{r} = \mathbf{e}_1x_1 + \mathbf{e}_2x_2 + \mathbf{e}_3x_3$. Note that the n-adic $(D^2)^{n/2}\mathbf{r}^n$ and $(n+1)$-adic $(D^2)^{(n+1)/2}\mathbf{r}^{n+1}$ are *constant* polyadics. The force polyadic for the slightly deformed sphere, Eq. (64), is (R5a)

$$
\begin{aligned}
{}_{n+2}\overset{(F)}{\mathfrak{K}}_O = \delta_{n2}\pi a^3\mathbf{II} + \varepsilon\frac{2^n 6\pi a^{n+1}}{(2n)!}\Big[\tfrac{1}{2}n(n-1)(2n-1)\mathbf{IC}_{n-2}\mathbf{I} \\
+ \mathbf{IC}_n - \frac{(n-1)(2n-3)}{2(2n+1)}\mathbf{C}_n\mathbf{I} - \frac{1}{(n+2)(2n+5)}\mathbf{C}_{n+2}\Big] + O(\varepsilon^2)
\end{aligned}
$$

valid for $n' \geqslant 1$. Here δ_{ij} is the Kronecker delta and $\mathbf{C}_m = \nabla^m(r^m f_m)$ is a constant m-adic. For the special case $n = 0$, ${}_2\overset{(F)}{\mathfrak{K}}_O$ is identical to $\overset{(t)}{\mathbf{K}}$ in Eq. (65). The origin O refers to the center of the undeformed sphere. By deleting the f_1-term the preceding formula may also be applied at the center of reaction of the slightly deformed sphere. A comparable formula is also available for the torque polyadic ${}_{n+2}\overset{(T)}{\mathfrak{K}}_O$ to the first order in ε (R5a).

It should be clearly understood that the formulas tabulated in the preceding paragraph apply only to the case where $\mathbf{u}$ is a Stokes flow.

For simple flow fields the polyadic resistance formulation normally provides more insight into the physical phenomena which arise than does the equivalent symbolic operator method. In the case of a general linear shear field the undisturbed flow at infinity may be written in the form (B23)

$$\mathbf{u} = \mathbf{u}_O + \boldsymbol{\omega}_f \times \mathbf{r} + \mathbf{S} \cdot \mathbf{r} \tag{106}$$

where O is an arbitrary point affixed to the particle; $\mathbf{u}_O$ is the approach velocity at O; $\mathbf{r}$ is measured from O;

$$\boldsymbol{\omega}_f = \tfrac{1}{2}\nabla \times \mathbf{u} \tag{107}$$

is the angular velocity of a fluid particle, and

$$\mathbf{S} = \tfrac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^\dagger] \tag{108}$$

is the rate of shear dyadic for the incompressible flow; $\boldsymbol{\omega}_f$ and $\mathbf{S}$ are constants, independent of choice of origin. The hydrodynamic force and torque about O are then

$$\mathbf{F} = -\mu\left[\overset{(t)}{\mathbf{K}} \cdot (\mathbf{U}_O - \mathbf{u}_O) + \overset{(c)}{\mathbf{K}}{}_O^\dagger \cdot (\boldsymbol{\omega} - \boldsymbol{\omega}_f) + \overset{(F)}{\boldsymbol{\mathfrak{S}}}_O : \mathbf{S}\right] \tag{109}$$

$$\mathbf{T}_O = -\mu\left[\overset{(c)}{\mathbf{K}}_O \cdot (\mathbf{U}_O - \mathbf{u}_O) + \overset{(r)}{\mathbf{K}}_O \cdot (\boldsymbol{\omega} - \boldsymbol{\omega}_f) + \overset{(T)}{\boldsymbol{\mathfrak{S}}}_O : \mathbf{S}\right] \tag{110}$$

where the constant triadic $\overset{(F)}{\boldsymbol{\mathfrak{S}}}_O$ and the pseudotriadic $\overset{(T)}{\boldsymbol{\mathfrak{S}}}_O$ are, respectively, the intrinsic *shear-force* and *shear-torque* resistance triadics at O (B23). They are related to the previous resistance polyadics by the expressions

$$\overset{(F)}{\boldsymbol{\mathfrak{S}}}_O = -{}_3\overset{(F)}{\boldsymbol{\mathfrak{R}}}_O - \tfrac{1}{2}\overset{(c)}{\mathbf{K}}{}_O^{\dagger} \cdot \boldsymbol{\varepsilon} \tag{111}$$

$$\overset{(T)}{\boldsymbol{\mathfrak{S}}}_O = -{}_3\overset{(T)}{\boldsymbol{\mathfrak{R}}}_O - \tfrac{1}{2}\overset{(r)}{\mathbf{K}}_O \cdot \boldsymbol{\varepsilon} \tag{112}$$

Equations (109)–(110) show that the force and torque on the particle are *linear* vector functions of the translational slip velocity $\mathbf{U}_O - \mathbf{u}_O$, the angular slip velocity $\boldsymbol{\omega} - \boldsymbol{\omega}_f$, and the shear rate $\mathbf{S}$.

For an ellipsoidal particle the two shear triadics take the following forms at its center (B23):

$$\overset{(F)}{\boldsymbol{\mathfrak{S}}}_R = \mathbf{0} \tag{113}$$

and

$$\begin{aligned}\overset{(T)}{\boldsymbol{\mathfrak{S}}}_R = \frac{8\pi}{3}\Bigg[&(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_1\mathbf{e}_3\mathbf{e}_2)\,\frac{a_3^{\,2} - a_2^{\,2}}{a_3^{\,2}\alpha_3 + a_2^{\,2}\alpha_2} \\ &+ (\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_1\mathbf{e}_3)\,\frac{a_1^{\,2} - a_3^{\,2}}{a_1^{\,2}\alpha_1 + a_3^{\,2}\alpha_3} \\ &+ (\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1)\,\frac{a_2^{\,2} - a_1^{\,2}}{a_2^{\,2}\alpha_2 + a_1^{\,2}\alpha_1}\Bigg]\end{aligned} \tag{114}$$

These may also be derived from Eqs. (96) and (97).

The corresponding formulas for the slightly deformed sphere, Eq. (64), are (B23)

$$\overset{(F)}{\mathfrak{S}}_R = \varepsilon \frac{2\pi}{7} a^2 \nabla\nabla\nabla(r^3 f_3) + O(\varepsilon^2) \tag{115}$$

and

$$\overset{(T)}{\mathfrak{S}}_R = \varepsilon 2\pi a^3 [\boldsymbol{\varepsilon} \cdot \nabla\nabla(r^2 f_2) + \{\boldsymbol{\varepsilon} \cdot \nabla\nabla(r^2 f_2)\}^\dagger] + O(\varepsilon^2) \tag{116}$$

Alternatively, in invariant form,

$$\overset{(F)}{\mathfrak{S}}_R = \frac{15}{8}\left(\frac{3S}{V}\right)^2 \int_S d\mathbf{S} \cdot \mathbf{rrrr} + O(\varepsilon^2) \tag{117}$$

$$\overset{(T)}{\mathfrak{S}}_R = \frac{5}{2}\left(\frac{3S}{V}\right)^2 \left[\boldsymbol{\varepsilon} \cdot \int_S \mathbf{rrr} \cdot d\mathbf{S} + \left(\boldsymbol{\varepsilon} \cdot \int_S \mathbf{rrr} \cdot d\mathbf{S}\right)^\dagger\right] + O(\varepsilon^2) \tag{118}$$

where $\mathbf{r}$ is measured from the centroid.

To illustrate the application of these relations, consider the force experienced by the ovoid body of revolution

$$r = a[1 + \varepsilon P_3(\cos\theta)]$$

when immersed in a uniform simple shearing flow with streamlines instantaneously parallel to its symmetry axis, the velocity being zero at its centroid (see Fig. 1). Here,

$$P_3(\cos\theta) = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

is the Legendre polynomial of order 3. This body lacks fore-aft symmetry. As stated following Eq. (68), the center of reaction of any slightly deformed sphere coincides with its centroid to at least $O(\varepsilon)$ (B21). Let the positive Rx_1 axis point from the blunted pole of the ovoid to its pointed end, as in Fig. 1, and set $\cos\theta = x_1/r$, where $\mathbf{r} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3$ is measured from R. This makes $\varepsilon \geqslant 0$. The uniform simple shearing flow is taken to be

$$\mathbf{u} = \mathbf{e}_1 x_3 S$$

with $S \geqslant 0$. The orthogonal triad of unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are to be right handed in their natural order and of such a nature that the positive Rx_2 axis points in the direction of the vorticity vector, $\nabla \times \mathbf{u}$. In the present case we have that $\mathbf{U}_R - \mathbf{u}_R = \mathbf{0}$ and, from Eq. (67), $\overset{(c)}{\mathbf{K}}_R = O(\varepsilon^2)$. Thus, according to Eq. (109), the body experiences a force

$$\mathbf{F} = -\mu \overset{(F)}{\mathfrak{S}}_R : \mathbf{S} + O(\varepsilon^2)$$

But

$$r^3 f_3 = r^3 P_3 = \tfrac{1}{2}(5{x_1}^3 - 3r^2 x_1)$$

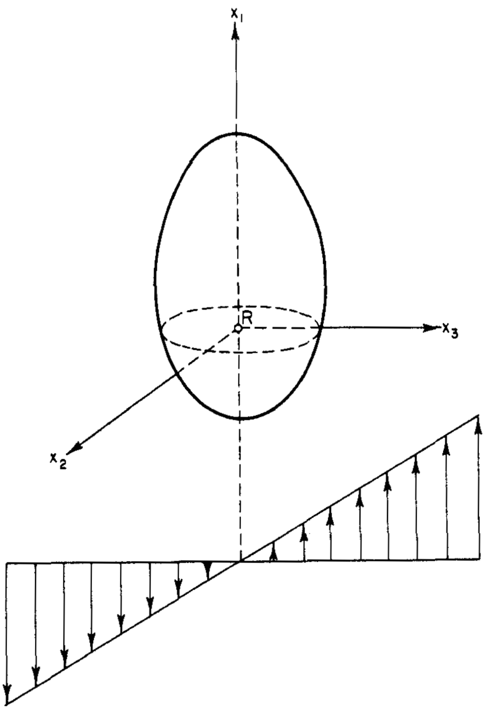

FIG. 1. Simple shear flow parallel to the axis of an ovoid body of revolution.

Hence, from Eq. (115),

$$\overset{(F)}{\mathfrak{S}}_R = \varepsilon \frac{6\pi}{7} a^2[5\mathbf{e}_1\mathbf{e}_1\mathbf{e}_1 - \{\mathbf{I}\mathbf{e}_1 + (\mathbf{I}\mathbf{e}_1)^\dagger + \mathbf{e}_1\mathbf{I}\}] + O(\varepsilon^2)$$

Furthermore, from Eq. (108),

$$\mathbf{S} = \tfrac{1}{2}S(\mathbf{e}_3\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3)$$

The body thus experiences a lift force

$$\mathbf{F} = \mathbf{i}_3 \frac{6\pi}{7} \varepsilon\mu a^2 S + O(\varepsilon^2)$$

at right angles to both the streamlines and vortex lines, in the sense shown in Fig. 1. This calculation agrees with a result of Bretherton (B30), obtained from a detailed solution of the equations of motion for the ovoid body.

A systematic investigation of the reduction in the number and type of independent components of the shear-force and torque triadics is presented by Brenner (B23) for particles possessing some common types of geometric symmetry. These properties might have been more simply obtained directly from the symmetry properties of the original symbolic force and torque operators. It is found, for example, that a body of revolution (about the Rx_1 axis, say) possessing fore-aft symmetry is characterized by a *single* scalar coefficient, in accordance with the relations $\overset{(F)}{\mathfrak{S}}_R = \mathbf{0}$ and

$$\overset{(T)}{\mathfrak{S}}_R = [\boldsymbol{\varepsilon} \cdot \mathbf{e}_1\mathbf{e}_1 + (\boldsymbol{\varepsilon} \cdot \mathbf{e}_1\mathbf{e}_1)^\dagger]\mathfrak{S} \tag{119}$$

where $\mathbf{e}_1$ is a unit vector parallel to the axis of revolution and $\mathfrak{S}$ is a scalar.

The contrast in behavior between the force and torque triadics depends on the fact that the former is a true tensor whereas the latter is a pseudotensor. If the particle under consideration is either spherically or helicoidally isotropic then $\overset{(\,)}{\mathfrak{S}}_R$ must be a scalar or pseudoscalar multiple of the alternator $\boldsymbol{\varepsilon}$, for the latter is the only isotropic third-rank tensor (A5). But, since $\mathbf{S}$ is a symmetric dyadic, $\boldsymbol{\varepsilon} : \mathbf{S}$ vanishes identically. Hence, neither the shear-force nor shear-torque triadics can contribute to the force or torque on the particle. Both may therefore be taken to be identically zero, in agreement with a more formal analysis (B23).

Though relations of the form (109) and (110) are merely special cases of the more general symbolic operator relations (81) and (82), they may also be regarded as phenomenological equations in their own right. The polyadic resistance coefficients appearing therein can, at least in principle, be determined experimentally from appropriate measurements of the hydrodynamic forces and torques for an appropriate number of orientations of the particle relative to the principal axes of dilatation of the fluid motion.

The hydrodynamic force and torque on a homogeneous, neutrally buoyant body vanish, providing that the motion is sufficiently slow. Equations (109) and (110) then yield for the translational and angular slip velocities (B23)

$$\mathbf{U}_O - \mathbf{u}_O = -\left[\left(\overset{(t)}{\mathbf{K}} - \overset{(c)}{\mathbf{K}}_O{}^\dagger \cdot \overset{(r)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(c)}{\mathbf{K}}_O\right)^{-1} \cdot \left(\overset{(F)}{\mathfrak{S}}_O - \overset{(c)}{\mathbf{K}}_O{}^\dagger \cdot \overset{(r)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(T)}{\mathfrak{S}}_O\right)\right] : \mathbf{S} \tag{120}$$

$$\boldsymbol{\omega} - \boldsymbol{\omega}_f = -\left[\left(\overset{(r)}{\mathbf{K}}_O - \overset{(c)}{\mathbf{K}}_O \cdot \overset{(t)}{\mathbf{K}}{}^{-1} \cdot \overset{(c)}{\mathbf{K}}_O{}^\dagger\right)^{-1} \cdot \left(\overset{(T)}{\mathfrak{S}}_O - \overset{(c)}{\mathbf{K}}_O \cdot \overset{(t)}{\mathbf{K}}{}^{-1} \cdot \overset{(F)}{\mathfrak{S}}_O\right)\right] : \mathbf{S} \tag{121}$$

The constant triadic and pseudotriadic in square brackets are intrinsic resistance coefficients. Furthermore, the latter is independent of choice of origin, as can be seen from the fact that $\boldsymbol{\omega} - \boldsymbol{\omega}_f$ and $\mathbf{S}$ are. These relations show that an anisotropic, neutrally buoyant particle immersed in a shear flow is, in general, unable to follow the local fluid motion. Rather, linear and angular slip must obtain.

It follows, upon substituting Eqs. (60) and (113) into (120), that a neutrally buoyant ellipsoidal particle is carried along with the flow; that is, $\mathbf{U}_R = \mathbf{u}_R$. Furthermore, as shown by Jeffery (J4), and as follows from integration of (121), a homogeneous ellipsoid of revolution in a uniform, simple shear flow moves in a closed, periodic orbit relative to an observer translating with the particle. Jeffery's theoretical calculation of the orbit is accurately confirmed by the experimental work of Mason and co-workers (F7, G8, M5) using spheres, circular disks, fused spherical doublets, and circular rods, though for long rods it was found necessary to introduce an "equivalent length" to account for the fact that the experimental period of rotation was only one-half to three-quarters that predicted by Jeffery's relationships. Bretherton (B30) has pointed out that the latter two bodies behave like ellipsoids of revolution with respect to their motion in a Couette flow. Based essentially on Eqs. (120) and (121), Bretherton (B30) has also shown that there exist exceptional shapes, including bodies of revolution of large aspect ratio, which behave very differently in such flows. They do not undergo periodic motion, but rather directly take up one of two opposite stable orientations and, remaining in that orientation, migrate across the streamlines towards the walls of the apparatus or to the center of the flow. Questions of permanent orientations of this type are utilized by Ericksen (E2) in connection with models of the non-Newtonian behavior displayed by "anisotropic fluids." Application of Eq. (121) to other models of non-Newtonian behavior is discussed by Giesekus (G4, G5a).

b. Liquid Droplets. The behavior of *deformable* bodies in shear and related flows is very different from that of rigid bodies. In particular, the shape of a neutrally buoyant immiscible liquid droplet immersed in a continuous liquid undergoing shear is not governed solely by the bulk and interfacial properties of the two phases, but depends also upon the rate of shear. Accordingly, the shape of the boundary must be determined simultaneously along with the detailed solution for the inner and outer fluid motions. Thus, despite the linearity of Stokes equations, the problem is intrinsically nonlinear. Upon expressing the equations of motion and boundary conditions in terms of non-dimensional variables, it is readily shown that the surface of the droplet is expressible in the following general form:

$$\mathbf{r}/a = \text{function}\,(\mu a \mathrm{S}/\gamma,\, \eta)$$

where $\mathbf{r}$ is the position vector of a point on the surface of the droplet (relative to, say, an origin at the centroid of the deformed drop), a is the radius of a sphere having the same volume as the deformed droplet, γ is the equilibrium interfacial tension, μ is the continuous-phase viscosity, and η is the ratio of droplet to continuous-phase viscosity. Because of the nonlinearity, the preceding functional relationship has never been established in its full generality.

Taylor (T1b, T2) has, however, obtained first-order perturbation solutions valid for two limiting cases.

For the case where the interfacial forces dominate those due to viscosity, Taylor (T2) found that the surface of the droplet is described by the equation of the slightly deformed sphere[13]

$$\frac{1}{a^2}\left(\mathbf{I} - \frac{4\mu a}{\gamma}\frac{19\eta + 16}{16\eta + 16}\mathbf{S}\right) : \mathbf{rr} + o(\varepsilon) = 1 \tag{122a}$$

for sufficiently small values of the deformation parameter $\varepsilon = \mu a\bar{S}/\gamma$. Here, $\bar{S} = (\mathbf{S}:\mathbf{S})^{1/2}$ is the intensity of the shear. To the first order in the deformation the droplet is therefore ellipsoidal, its principal axes coinciding with the principal axes of dilatation, i.e., of $\mathbf{S}$. To this approximation the departure from the spherical shape is *proportional to the rate of strain*. Equation (122a) applies to any incompressible flow, provided that $\mathbf{S}$ is interpreted as the *local* rate of strain of the unperturbed flow (C2, C3); that is, the shear need not be uniform. For a simple uniform shear flow the principal axes of dilatation are inclined at an angle of 45° with respect to the streamlines; hence, the principal axes of the droplet are equally inclined relative to the streamlines. Equation (122a) was experimentally confirmed for Couette and hyperbolic shear flows by Taylor (T2) and, more accurately, by Rumscheidt and Mason (R9) for $\eta < 10$ (approx.), and by Goldsmith and Mason (G9) for Poiseuille flows. For values of the deformation parameter ε greater than about 0.2, significant departure from the linear relation (122a) was observed in most cases. The droplets burst at about $\varepsilon = 0.5$, in good agreement with Taylor's (T1b) qualitative prediction, this being approximately the point at which the viscous forces tending to stretch the drop exceed the interfacial forces tending to maintain its integrity. Beyond the effective linear range the drop shape depends nonlinearly on the shear $\mathbf{S}$. The shape of the drop just prior to bursting is therefore different for different nonuniform shear flows (and, of course, for different viscosity ratios). Rumscheidt and Mason (G9) noted a large variety of differently shaped drops prior to bursting.

Taylor (T2) also treated the other extreme case where $\varepsilon \to \infty$, but only for the special case of the "very viscous drop"— that is, $\eta^{-1} \to 0$. When ε is large the viscous forces greatly exceed the interfacial forces, and Taylor ignores the latter as a first approximation. The shape of the droplet is then only a function of the viscosity ratio. For a simple uniform shear, Taylor finds, on

[13] In polar form the equation of this slightly deformed sphere is

$$r = a\left[1 + \frac{2\mu a}{\gamma}\frac{19\eta + 16}{16\eta + 16}\frac{\mathbf{S}:\mathbf{rr}}{r^2} + o\left(\frac{\mu a\bar{S}}{\gamma}\right)\right]$$

the assumption that the droplet departs but little from the spherical shape, that the droplet is ellipsoidal—its major axis being parallel to the streamlines of the unperturbed flow. The equation of the droplet surface projected onto the plane containing the shear is

$$r/a = 1 + \tfrac{5}{4}\eta^{-1} \cos 2\phi + o(\eta^{-1}) \tag{122b}$$

where the angle $\phi = 0°$ lies parallel to the direction of the streamlines. The experiments of Taylor (T2) and Rumscheidt and Mason (R9) only approximately confirmed this result at the larger η values [$\eta > 5$ (approx.)] in the linear range. Above this range the droplets did not burst, but rather were elongated into long filaments in the Couette apparatus. These droplets did, however, burst in the "four-roller" apparatus, where the shear was hyperbolic rather than uniform.

Though agreement with experiment seems rather good, further theoretical analysis is obviously required to fix the ranges of validity of Eqs. (122a) and (122b). This is especially true in the very viscous droplet problem. One is dealing here with a two-parameter problem; and as is true for all such problems, the range of applicability is not fixed merely by requiring that the two parameters ε^{-1} and η^{-1} each be small. For the quantity $\varepsilon^{-1}/\eta^{-1}$ (or some variation thereof) may then be either small or large, depending upon experimental circumstances; and it seems highly unlikely that one formula could correctly describe all possible limiting cases. It thus appears that Taylor's analysis applies only to the case where $\varepsilon^{-1} \ll \eta^{-1} \ll 1$.

3. *Wall-Effects*

Though wall-effects are rarely important in industrial-size equipment they are important in small-scale laboratory apparatus. Indeed, because of the relatively large wall-effects in Stokes flows it is difficult to perform experiments which are free of such effects. A sphere falling along the axis of circular tube whose cross sectional area is 100 times that of the sphere settles at less than 75 percent of the velocity it would achieve in an unbounded fluid. Even with a 10,000 : 1 cross sectional area ratio the settling velocity is decreased by more than 2 percent. As a consequence, wall-effects have been extensively studied in the Stokes regime. In this section we shall examine some general principles and results pertaining to such effects. Attention is again restricted to quasi-static motions.

Consider a solid particle of arbitrary shape settling in a container of any shape which partially or completely bounds the fluid externally. If it does not wholly bound the fluid, it is assumed in the subsequent discussion either that the fluid extends to an effectively infinite distance from the particle (e.g., an infinite or semi-infinite circular cylinder) or else that a planar free surface exists at a finite distance from the body. In short, the general theory depends upon the condition that no work is done by the fluid stresses at the apparatus boundaries. Under these circumstances Eqs. (38) and (39) continue to describe the instantaneous hydrodynamic force and torque on a particle moving through an otherwise quiescent fluid (B22). The three resistance dyadics remain independent of μ, $\mathbf{U}_O$, and $\boldsymbol{\omega}$. However, they are no longer intrinsic properties of the particle alone, but now also depend upon the geometrical configuration of the boundary wetted by the fluid, as well as upon the instantaneous orientation and position of the particle relative to these boundaries. They are, nevertheless, purely geometric parameters. With this reservation the entire general theory outlined in Section II,C,1 up to and including Eq. (55), remains valid, providing that we supplement the definition of the intrinsic translational and rotational fields with the additional boundary conditions appropriate to the fluid at the container boundaries. In the presence of boundaries, the point at which the coupling dyadic is symmetric does not generally coincide with the center of reaction of the particle in an unbounded fluid.

There exist few particle-boundary combinations for which the *particle-boundary resistance dyadics* in Eqs. (38) and (39) are known for all physically possible particle-to-wall dimensions. They are known, for example, for the trivial case of a spherical particle at the center of a concentric spherical boundary, the space between them being filled with fluid.[14] The translation and rotation dyadics for this case are clearly isotropic, while the coupling dyadic obviously vanishes at this common center.

Fortunately, in most practical investigations, such complete information is unnecessary. Rather, it usually suffices to know only certain components of these dyadics, and then only in limiting cases. If a/l represents a characteristic particle-to-wall dimension ratio, these limiting cases correspond to the extreme cases where a/l is either very small or very near unity. In the former case the method of "reflections" (cf. H9) provides a useful technique for obtaining the wall correction. In the latter case, corresponding to the situation where the particle is extremely close to the wall, lubrication-theory type approximations (B7, B29, C11, D7, G5d, H15, K8, M9, M10, S8) normally suffice to obtain the required correction. Intermediate cases are rarely of interest.

As a case in point consider a spherical particle of radius a at the axis of an infinitely long circular cylinder of radius l. If we choose the origin (R) at the sphere center and let $\mathbf{e}_1$ be a unit vector parallel to the cylinder axis,

[14] The force on the inner sphere for the translational case is given independently by Cunningham (C21), Haberman and Sayre (H3), and Williams (W8), though the first-mentioned reference contains a typographical error. The torque on the inner sphere for the rotational case is given by Lamb (L5, p. 589).

it is clear on symmetry grounds that

$$\overset{(t)}{\mathbf{K}} = \mathbf{e}_1\mathbf{e}_1\overset{(t)}{K}_{\parallel} + (\mathbf{I} - \mathbf{e}_1\mathbf{e}_1)\overset{(t)}{K}_{\perp} \tag{123}$$

$$\overset{(r)}{\mathbf{K}}_R = \mathbf{e}_1\mathbf{e}_1\overset{(r)}{K}_{\parallel} + (\mathbf{I} - \mathbf{e}_1\mathbf{e}_1)\overset{(r)}{K}_{\perp} \tag{124}$$

$$\overset{(c)}{\mathbf{K}}_R = \mathbf{0} \tag{125}$$

The components of primary interest are the parallel components, $\overset{(t)}{K}_{\parallel}$ and $\overset{(r)}{K}_{\parallel}$, for these give the force and torque on the sphere when it translates along the axis and rotates about this axis, respectively.

In the translational case the method of reflection yields (see B8, H3, and references to earlier papers given therein), to the lowest degree of approximation,

$$\overset{(t)}{K}_{\parallel} = \frac{6\pi a}{1 - 2.10444(a/l) + O(a/l)^3} \tag{126}$$

for small a/l. For a/l near unity an elementary lubrication-theory treatment yields (M10)

$$\overset{(t)}{K}_{\parallel} = 6\pi a \frac{3\pi\sqrt{2}/8}{(1 - a/l)^{5/2}} \tag{127}$$

The former of these results agrees quite accurately with the "exact" solution (B8, H3) and with experimental data up to $a/l \approx 0.2$ (cf. H3). Equation (127) also agrees well with experiment for $a/l > 0.95$ (M10).

Similarly, in the rotational case, we have in the two extreme limits (B29, H1)

$$\overset{(r)}{K}_{\parallel} = \frac{8\pi a^3}{1 - 0.79682417(a/l)^3 + O(a/l)^{10}} \tag{128}$$

and, as $a/l \to 1$ (B29),[15]

$$\overset{(r)}{K}_{\parallel} = 8\pi a^3 \frac{\pi}{2(a/l)^2}\left[\frac{1}{\{1 - (a/l)^2\}^{1/2}} - 1\right] \tag{129}$$

These both agree remarkably well with the "exact" solution (B29), as can be

[15] This is an even better approximation than the one derived in the original paper (B29). It is obtained by utilizing the velocity field generated by the relative rotation of two coaxial circular cylinders of radii a and l, rather than utilizing the velocity field generated by the shearing motion of two parallel planes separated by a distance $l - a$, as was done originally. D. L. Ripps pointed out this improvement to me.

seen from Table I, in which is tabulated the ratio of the torque T on the sphere to the torque T_∞ in the unbounded fluid under otherwise identical conditions.

TABLE I

WALL-EFFECT FOR THE TORQUE ON A SPHERE ROTATING ABOUT THE AXIS OF A CIRCULAR CYLINDER[a]

	T/T_∞		
a/l	Exact solution, Ref. (B29)	First reflection approximation, Eq. (128)	Lubrication theory approximation, Eq. (129)
0	1	1	—
0.1	1.0007975	1.0007975	—
0.2	1.0064155	1.0064154	—
0.3	1.0219877	1.0219872	—
0.4	1.0537442	1.053737	—
0.5	1.1106942	1.11062	—
0.6	1.2084380	1.2078	—
0.7	1.3795032	1.376	—
0.8	1.7101933	1.69	—
0.85	2.0145515	1.96	—
0.9	2.5567226	—	2.51
0.95	3.8610279	—	3.83
0.975	5.7962397	—	5.784
0.99	9.75569	—	9.758
0.995	14.288	—	14.299
0.999	$33._5$	—	33.63
1	∞	—	∞

[a] $|T_\infty| = 8\pi\mu a^3\omega$.

Similar agreement exists for the quasi-steady approach of a sphere towards a rigid plane wall. If F is the force on the sphere of radius a, and h is the distance from the sphere center to the wall, the exact solution, valid for all a/h, is (B14, M6)

$$\frac{|F|}{6\pi\mu a U} = \sigma = \frac{4}{3}\sinh\alpha \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \times \left[\frac{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha}{4\sinh^2(n+\frac{1}{2})\alpha - (2n+1)^2\sinh^2\alpha} - 1\right] \tag{130}$$

where the parameter α is defined as

$$\cosh\alpha = h/a \tag{131}$$

In the limit $a/h \to 0$, a straightforward expansion of the hyperbolic functions for large values of α yields

$$\sigma = \frac{1}{1 - (9/8)(a/h) + O(a/h)^3} \tag{132}$$

This accords with the original result of Lorentz (L11), obtained by employing a first reflection.[16] In the opposite case, where the sphere is very near the wall, i.e., $a/H \to \infty$ (where $H = h - a$), one finds from (131) that $\alpha^2 \to 2H/a$. Upon expansion of all the hyperbolic functions in (130) for small α, one obtains

$$\sigma \to \frac{64}{\alpha^2} \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)^2(2n+1)(2n+3)^2}$$

The sum over n may be performed by the following artifice. By identity

$$\frac{n(n+1)}{(2n-1)^2(2n+1)(2n+3)^2} = \frac{1}{64}\left[\frac{1}{(2n-1)(2n+1)} - \frac{1}{(2n+1)(2n+3)}\right] + \frac{3}{128}\left[\frac{1}{(2n-1)^2} - \frac{1}{(2n+3)^2}\right]$$

But

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} - \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right) - \left(\frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right) = \frac{1}{1\cdot 3}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n+3)^2} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) - \left(\frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) = \frac{1}{1^2} + \frac{1}{3^2}$$

Hence, when the sphere is very near the wall

$$\sigma \to a/H \tag{133a}$$

This result is identical to that obtained from a conventional lubrication-theory approximation (M1), based on the flow field generated when two parallel plane approach one another (L6). The general formula (130) is in excellent agreement, over the entire a/h range, with the experimental results of MacKay *et al.* (M1, M2).

[16] Lorentz's general scheme for solving boundary-value problems involving plane walls has been rederived by Maude (M7), apparently unaware of the original work.

Conventional lubrication-theory methods may be regarded as furnishing only the leading term in an asymptotic expansion of the exact solution of the problem. Progress has recently been made (C19) in the formulation of a general theory leading to higher-order terms in the expansion. Singular perturbation methods requiring the formation of inner and outer expansions (cf. Section III,C) are involved, the inner expansion being valid near the point of contact between the particle and wall. The results take the form of an asymptotic expansion in a small, nondimensional clearance parameter. For example, for a sphere approaching a plane wall, Eq. (133a) is replaced by

$$\sigma = \left(\frac{H}{a}\right)^{-1}\left[1 - \frac{1}{5}\frac{H}{a}\ln\frac{H}{a} + \mathrm{O}\left(\frac{H}{a}\right)\right] \tag{133b}$$

if the quasi-static Stokes equations are regarded as the governing equations of motion. This equation is asymptotically valid as $H/a \to 0$. Small inertial effects may also be incorporated into the singular perturbation analysis (C19), in which case the solution takes the form of a double expansion in two small parameters—a dimensionless gap width and an appropriate Reynolds number based on gap width. Thus, when the complete Navier–Stokes equations (including *unsteady-state* terms) are applied to the motion of a sphere towards a plane wall, Eq. (133b) is replaced by

$$\sigma = \left(\frac{H}{a}\right)^{-1}\left[1 - \left(\frac{1}{5}\frac{H}{a} + \frac{19}{5}R\right)\ln\frac{H}{a} + \cdots\right] \tag{133c}$$

where $R = HU\rho/\mu$. The above expansion is valid for $H/a \ll 1$ and $R \ll 1$, it further being supposed that $R(H/a)^{-1} = aU\rho/\mu$ is of order unity with respect to the parameter H/a; that is, the solution is asymptotically valid in the limit as $H/a \to 0$ with $aU\rho/\mu$ fixed.

The validity of Eq. (133b) [including an explicit calculation of the coefficient ($= 0.804597$) of the term of $\mathrm{O}(H/a)$ within the brackets] was independently confirmed (C19) by including higher-order terms in the expansion of Eq. (130) leading to Eq. (133a). The resulting sum over n is itself highly singular, and requires the formation of "inner" and "outer" portions; for no matter how small α may be, $n\alpha$ will not be small for sufficiently large n.

When a/l is small it is possible to obtain a general expression for the effect of boundaries on the quasi-static force and torque experienced by a translating-rotating particle of arbitrary shape in a fluid which is otherwise at rest (B16, B20, C20). This relation takes the form

$$\begin{Bmatrix} \mathbf{F} \\ \mathbf{T}_O \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^\infty \\ \mathbf{T}_O{}^\infty \end{Bmatrix} + \begin{Bmatrix} \overset{(t)}{\mathbf{K}}{}^\infty \\ \overset{(c)}{\mathbf{K}}{}_O{}^\infty \end{Bmatrix} \cdot \mathbf{W}_O \cdot \mathbf{F}^\infty + \mathrm{O}\left(\frac{a}{l}\right)^2 \qquad \begin{matrix}(134a)\\(134b)\end{matrix}$$

Here, $\mathbf{F}^{\infty}$ and $\mathbf{T}_O{}^{\infty}$ refer to the force and torque (about the arbitrary point O) experienced by the particle in the *unbounded* fluid when it translates and rotates with the same velocities $\mathbf{U}_O$ and $\boldsymbol{\omega}$ as in the bounded fluid, the orientation of the particle being the same for the two cases. The quantities $\overset{(t)}{\mathbf{K}}{}^{\infty}$ and $\overset{(c)}{\mathbf{K}}_O{}^{\infty}$ are the translation and coupling dyadics for the particle in the unbounded fluid. Also, $\mathbf{W}_O$ is the wall-effect dyadic (B20, C20) evaluated at the point in the fluid presently occupied by O. In general, $\mathbf{W}_O$ is a constant, *symmetric* dyadic which depends only upon the size and shape of the boundaries and upon the location of O relative to the bounding walls. In particular, $\mathbf{W}_O$ is independent of μ, $\mathbf{U}_O$, $\boldsymbol{\omega}$, of the size and shape of the particle, and of the orientation of the particle relative to the boundaries. At each point O in space it is, therefore, an intrinsic geometric property of the container boundaries alone. Inasmuch as $\overset{(t)}{\mathbf{K}}{}^{\infty} = \mathrm{O}(a)$, $\overset{(c)}{\mathbf{K}}_O{}^{\infty} = \mathrm{O}(a^2)$, $\mathbf{W}_O = \mathrm{O}(l^{-1})$ and $\mathbf{T}_O{}^{\infty} = \mathrm{O}(aF^{\infty})$, Eq. (134) furnishes an expression for the wall-effect correctly to terms of $\mathrm{O}(a/l)$. Thus, the significance of Eq. (134) lies in the fact that to this order the wall correction factor is divided into *separate* contributions from the particle and boundary. Equations (134a)–(134b) are equivalent to the three relations

$$
\begin{aligned}
\overset{(t)}{\mathbf{K}} &= \overset{(t)}{\mathbf{K}}{}^{\infty} + \overset{(t)}{\mathbf{K}}{}^{\infty} \cdot \mathbf{W}_O \cdot \overset{(t)}{\mathbf{K}}{}^{\infty} + \mathrm{O}(a/l)^2 \\
\overset{(r)}{\mathbf{K}}_O &= \overset{(r)}{\mathbf{K}}_O{}^{\infty} + \overset{(c)}{\mathbf{K}}_O{}^{\infty} \cdot \mathbf{W}_O \cdot \overset{(c)}{\mathbf{K}}{}_O^{\infty\dagger} + \mathrm{O}(a/l)^2 \qquad (134c) \\
\overset{(c)}{\mathbf{K}}_O &= \overset{(c)}{\mathbf{K}}_O{}^{\infty} + \overset{(c)}{\mathbf{K}}_O{}^{\infty} \cdot \mathbf{W}_O \cdot \overset{(t)}{\mathbf{K}}{}^{\infty} + \mathrm{O}(a/l)^2
\end{aligned}
$$

where the $\mathbf{K}$'s without the superscript ∞ refer to the values in the bounded fluid. Note that $\overset{(t)}{\mathbf{K}}$ and $\overset{(r)}{\mathbf{K}}_O$ are symmetric, as they should be.

For some commonly occurring types of geometric symmetry, Eq. (134a) applied to a body translating without rotation can be expressed in the more accurate form

$$\mathbf{F} = \left[\mathbf{I} - \overset{(t)}{\mathbf{K}}{}^{\infty} \cdot \mathbf{W}_O\right]^{-1} \cdot \mathbf{F}^{\infty} + \mathrm{O}(a/l)^3 \qquad (134d)$$

which is equivalent to the relation

$$\overset{(t)}{\mathbf{K}} = \left[\mathbf{I} - \overset{(t)}{\mathbf{K}}{}^{\infty} \cdot \mathbf{W}_O\right]^{-1} \cdot \overset{(t)}{\mathbf{K}}{}^{\infty} + \mathrm{O}(a/l)^3 \qquad (134e)$$

These more accurate formulas apply irrespective of the shape of the particle if, for example, the boundary possesses three mutually perpendicular symmetry planes intersecting at O. This occurs if the (point O fixed in the) particle translates along the axis of a circular cylinder or midway between two parallel planes (W8a). The more accurate formulas are also applicable to completely asymmetric boundaries if, for example, the particle possesses three mutually planes of reflection symmetry intersecting at O, e.g., a sphere or ellipsoid with center at O (W8a).

By way of example, the wall-effect dyadic for a solid plane wall bounding a semi-infinite fluid domain is (B20)

$$\mathbf{W}_O = [\mathbf{e}_1\mathbf{e}_1 2 + (\mathbf{I} - \mathbf{e}_1\mathbf{e}_1)] \frac{9}{16l}$$

where $\mathbf{e}_1$ is a unit vector normal to the wall and l is the perpendicular distance from the wall to the point O fixed in the particle. The comparable expression for a planar *free surface* is (B20)

$$\mathbf{W}_O = [\mathbf{e}_1\mathbf{e}_1 2 - (\mathbf{I} - \mathbf{e}_1\mathbf{e}_1)] \frac{3}{8l}$$

The utility of (134a) is most simply seen by considering the highly symmetrical case where one of the principal axes of translation of the particle lies parallel to a principal axis of the wall dyadic. In this case (134a) is equivalent to the scalar relation (B16)

$$\frac{F}{F_\infty} = \frac{1}{1 - \underline{W}_O \dfrac{|F_\infty|}{6\pi\mu U l} + o\left(\dfrac{a}{l}\right)} \tag{135}$$

where F is the hydrodynamic force on the particle in the bounded medium when it moves with speed U, and F_∞ is the force on the particle when it moves with the same speed and orientation (relative to $\mathbf{U}$) in the unbounded medium. The nondimensional scalar $\underline{W}_O$ is one of the *specific*, normalized eigenvalues of $\mathbf{W}_O$. This scalar has been made into an intensive property, independent of the *size* of the boundaries, by setting $\underline{W}_O = lW_O$. The wall dimension l may be chosen arbitrarily. The extraneous factor of 6π is included merely to give the equation a simple form for spherical particles. In the important case of a particle translating along the axis of a long circular cylinder of radius l, $\underline{W}_O = 2.10444$, as follows from Eq. (126) by observing that $|F_\infty|/6\pi\mu U = a$ for a spherical particle of radius a. An independent proof for this case is given by Chang (C4).

In general, the error term in Eq. (135) will be $O(a/l)^2$. However, if the prevailing symmetry is such that Eq. (134d) applies, the error will only be $O(a/l)^3$.

Values of $\underline{W}_O$ for other boundaries and various particle orientations relative to these boundaries are tabulated by Brenner (B16). In particular, for a particle falling at a fractional distance $\beta = b/l$ (b = distance from center of reaction of particle to cylinder axis) from the axis of a long circular cylinder, $\underline{W}_O = f(\beta)$ (B28).[17] For $\beta \to 0$

$$f(\beta) = 2.10444 - 0.6977\beta^2 + O(\beta^4) \tag{136}$$

and for $\beta \to 1$ (F1)

$$f(\beta) \to \frac{9}{16(1-\beta)} \tag{137}$$

provided that a/l and $(1-\beta)^{-1}a/l$ are both small compared with unity. The coefficient 9/16 arises from Lorentz's (L11) formula for the fall of a sphere parallel to a *plane* wall when the sphere is small compared with its distance from the wall. Famularo (F1) has evaluated $f(\beta)$ for intermediate values of β with results shown in Table II. Rather remarkably, $f(\beta)$ does not change monotonically with β but possesses a minimum at $\beta \approx 0.40$. This then is the point at which a sufficiently small particle would settle with its greatest speed.

TABLE II

DRAG ON AN ECCENTRICALLY POSITIONED PARTICLE IN A CIRCULAR CYLINDER; TABULATION OF $f(\beta)$ VS β

β	$f(\beta)$	β	$f(\beta)$
0	2.10444	0.40	2.04401
0.01	2.10436	0.41	2.04407
0.03	2.10381	0.43	2.04530
0.05	2.10270	0.45	2.04825
0.10	2.09763	0.50	2.06566
0.20	2.07942	0.60	2.16965
0.30	2.05691	0.70	2.45963
0.35	2.04805	0.80	3.2316
0.37	2.04567	0.90	5.905
0.39	2.04426	1	∞

In connection with terminal settling velocity experiments at the axis of a long circular cylinder, Eq. (135) may be written in the equivalent form

$$\frac{U_\infty}{U} = 1 + 2.10444 \frac{|F|}{6\pi\mu U l} + o\left(\frac{a}{l}\right) \tag{138}$$

where U_∞ and U are, respectively, the terminal settling speeds of the particle in the unbounded and bounded fluids for the same force F. Usually, one can

[17] Though Brenner and Happel (B28) did not consider the case where the particle rotated as it fell, the contribution of such rotation to the drag on the particle can be shown to be of smaller order than $O(a/l)$ (B20); hence, it is negligible to the present degree of approximation.

measure directly the weight $|F|$ of the particle corrected for the buoyancy of the fluid. From a *single* measurement of the settling velocity U in the cylinder one can therefore calculate what speed the same particle would have attained in the *unbounded* fluid, the orientation of the body relative to the direction of the gravitational field being the same in both cases. Equation (138) is in excellent agreement with a large body of experimental data (B16), and holds quite accurately even for a/l as large as 0.2, a being one-half the maximum linear dimension of the particle.

A formula comparable to Eq. (135) holds for the wall-effect experienced by rotating bodies (B16, B20). In the case of an axially symmetric body rotating symmetrically about the axis of a circular cylinder of radius l, the general formula reduces to

$$\frac{T}{T_\infty} = \frac{1}{1 - 0.79682417 \dfrac{|T_\infty|}{8\pi\mu\omega l^3} + o\left(\dfrac{a}{l}\right)^3} \tag{139}$$

where T and T_∞ are the respective torques in the bounded and unbounded fluids for the same angular speed ω. [Compare with (128) for a spherical particle, bearing in mind that $T_\infty = -8\pi\mu a^3\omega$.] Since $T_\infty = O(\mu\omega a^3)$ it follows that for small a/l the wall-effect is of $O(a/l)^3$. This is much less than for translating particles, where Eq. (135) shows the effect to be of $O(a/l)$. In general, Eq. (139) holds quite well, even for a/l as large as 0.5 (B29, B25).

Formulas of the type (135) apply even when more than one type of surface bounds the fluid. For a particle at the axis of a semi-infinitely long circular cylinder bounded below by the planar container bottom, Tanner (T1a) gives values of $\underline{W}_O$ as a function of h_1/l (h_1 = distance from particle to container bottom) based on an approximate, numerical solution of Stokes equations. In a similar context, an exact solution, obtained by the method of reflections, is available for the axially symmetric rotation of a small particle in a circular cylinder bounded below by the rigid container bottom and above by the planar free surface of the liquid (B25). The torque correction is presented in the form of Eq. (139) with the coefficient 0.7968 replaced by a more general rotational wall coefficient, *function*$(h_1/l, h_2/l)$, h_1 and h_2 being measured from the center of the particle to the container bottom and free surface, respectively. Results are tabulated for the complete ranges $0 < h_1/l < \infty$, $0 < h_2/l < \infty$. The techniques employed in solving this rotational wall-effect problem have recently been applied by Sonshine, Cox, and Brenner (S16) to the case of a nonskew particle *translating* parallel to a principal axis of translation along the axis of a circular cylinder of finite length possessing both a rigid bottom and a free surface. In addition to generalizing the results of

Tanner (T1a) the treatment is analytical rather than numerical. The results having important applications in falling-ball viscometry.

An extensive discussion and bibliography of further references to conventional wall-effects will be found in Happel and Brenner (H9). To this list should be added the theoretical study by Dean and O'Neill (D4) of the rotation of a sphere about an axis parallel to a nearby plane wall in an otherwise unbounded fluid, and a companion study by O'Neill (O2a) of the translation of a sphere parallel to a plane wall. Bipolar coordinates were employed to obtain exact solutions of the Stokes equations. These studies are particularly interesting in view of the fact that, due to the asymmetry of the flow, the rotating sphere experiences a force parallel to the wall whereas the translating sphere experiences a torque about the sphere center parallel to the wall. According to the remarks made in the second paragraph of Section II,C,3, these *cross effects* may be expressed in terms of the coupling dyadic in Eqs. (38) and (39) (where O refers to the sphere center). The Dean–O'Neill investigation permits computation of $\overset{(c)}{\mathbf{K}}_O$ from Eq. (38), whereas the O'Neill study permits a comparable computation via Eq. (39). Though these independent methods should have yielded the same value for the coupling dyadic, they did not. This discrepancy prompted a detailed check (G5d) of their calculations, which revealed a numerical error in the Dean–O'Neill (D4) paper. After rectifying the error it was found that the two independent computations of $\overset{(c)}{\mathbf{K}}_O$ did indeed agree.

The sphere-plane wall configuration represents one of the few nontrivial cases for which the particle-boundary resistance dyadics in Eqs. (38) and (39) are completely known. If $\mathbf{e}_1$ is a unit vector normal to the plane then the translation and rotation dyadics are given by equations of the form (123) and (124), provided that R is replaced by O—the sphere center; $\overset{(t)}{K}_{\parallel}$ is given as a function of a/h [see Eqs. (130)–(131)] by Brenner (B14) and Maude (M6); $\overset{(t)}{K}_{\perp}$ by O'Neill (O2a); $\overset{(r)}{K}_{\parallel}$ by Jeffery (J3); and $\overset{(r)}{K}_{\perp}$ by Goldman, Cox, and Brenner (G5d) based on their corrected version of the Dean–O'Neill (D4) treatment. The comparable expression for the coupling dyadic at the sphere center is

$$\overset{(c)}{\mathbf{K}}_O = \boldsymbol{\varepsilon} \cdot \mathbf{e}_1 \overset{(c)}{K}$$

where $\boldsymbol{\varepsilon}$ is the isotropic triadic. The scalar $\overset{(c)}{K}$ is given as a function of a/h by O'Neill (O2a) and by Goldman, Cox, and Brenner (G5d).

With regard to wall effects in fluids undergoing net flow, Goldman, Cox, and Brenner (G5e), using bipolar coordinates, obtained an exact solution to the problem of a neutrally buoyant sphere near a single plane wall in a semi-infinite fluid undergoing simple shear. In an unbounded fluid the translational velocity of the sphere center and angular velocity of the sphere are $U_O = Sh$ and $\omega = \frac{1}{2}S$, respectively, where S is the shear rate and h the distance from the wall to the sphere center. Corrections to these results due to the wall are given in terms of a/h. The method of solution is of interest in that the calculations of the force and torque were performed without having to solve the corresponding boundary-value problem. Rather, Eqs. (78)–(79) were employed, utilizing the rotational stress triadic $\overset{(r)}{\mathbf{\Pi}}_O$ gleaned from the bipolar coordinate solutions of Dean and O'Neill (D4; see the corrections to the latter given in Ref. G5d) and Jeffery (J3) for the rotation of a sphere near a plane wall in a quiescent fluid, and the translational stress triadic $\overset{(t)}{\mathbf{\Pi}}$ obtained from the comparable translational solutions given by O'Neill (O2a), and by Brenner (B14) and Maude (M6). The limiting case where the sphere approaches the wall ($a/h \to 1$) is independently treated via a lubrication-theory analysis.

Calculation of wall-effects when the fluid undergoes net flow, as in Poiseuille flow through a long tube open at both ends and containing a suspended particle, may be carried out by utilizing Eqs. (81) and (82). These symbolic operator equations continue to be valid, even in the presence of rigid bounding walls (see the remarks in the second paragraph of the present subsection—which are pertinent here too), providing that the quasi-static Stokes approximation is applicable. For a nonscrew-like body (i.e., $\overset{(c)}{\mathbf{K}}_R{}^{\infty} = \mathbf{0}$), or one which is not rotating, the hydrodynamic force on any solid particle is (B20)

$$\mathbf{F} = -\mu\overset{(t)}{\mathbf{K}} \cdot (\mathbf{U}_R - \mathbf{u}_R) + o(a/l) \tag{140}$$

where $\overset{(t)}{\mathbf{K}}$ is given to the first order in a/l by Eq. (134c); $\mathbf{u}_R$ refers to the value of the unperturbed flow field (e.g., the Poiseuille field) at the center of reaction of the particle. When sufficient particle-boundary symmetry is involved, this reduces to the same form as (135), the only modification required being the replacement of F_∞ in the left-hand side by F_∞', the infinite-medium force based on the *approach velocity* $\mathbf{u}_R$ (B16); i.e., if $U \neq 0$,

$$F_\infty' = F_\infty\left[1 - \frac{\mathbf{u}_R \cdot \mathbf{U}}{U^2} + o(a/l)\right] \tag{141}$$

In the case of Poiseuille flow in a circular tube, $\mathbf{u}_R = 2\mathbf{V}_m(1 - \beta^2)$. The vector $\mathbf{V}_m$ is equal in magnitude to the superficial velocity and points in the direction of net flow.

In the special case of a spherical particle moving parallel to the axis of a circular tube the force on the particle is (B28)

$$\mathbf{F} = -6\pi\mu a\left[\frac{\mathbf{U}_R - 2\mathbf{V}_m(1 - \beta^2) + (4/3)\mathbf{V}_m(a/l)^2}{1 - f(\beta)(a/l)} + O(a/l)^3\right] \tag{142}$$

which is a higher-order approximation then is furnished by (140). The numerator arises from Faxén's law, Eq. (89a), where the unperturbed flow $\mathbf{u}$ corresponds to Poiseuille flow in the tube. Equation (142) is applicable only when the sphere is not too near to wall; that is, when

$$1 \gg \frac{a}{l-b} = \frac{a}{l}\left(\frac{1}{1-\beta}\right)$$

This condition will always be met for sufficiently small a/l, irrespective of how near β is to unity. Equation (142) applies even if the sphere is rotating, for the additional wall-effect arising from the rotation is smaller than $O(a/l)^2$. Equation (142) is in excellent agreement with the experimental data of Fayon and Happel (F4) for Poiseuille flow past rigidly supported spherical particles in a long tube.

There exists a rather remarkable type of wall-effect which, rather than vanishing, approaches a finite limit as $a/l \to 0$—that is, as the fluid becomes "unbounded" relative to the particle. When a particle settles in an otherwise quiescent fluid confined within a vertical duct of constant cross section, a dynamic pressure difference ΔP^+ is set up, the pressure being greatest on that end of the duct towards which the particle is moving. The vector ΔP^+ points in the direction of diminishing pressure. It is natural to expect that when $a/l \to 0$ the vertical container walls can play no role. In this event, elementary momentum principles require that the external pressure-drop force, $\Delta P^+ A$, exerted on the fluid be exactly equal to $\mathbf{F}$, where A is the cross sectional area of the duct and $\mathbf{F}$ is the hydrodynamic force on the particle (equal and opposite to the net gravity force on the latter). Detailed theoretical analysis (B13, B17, B28) reveals that such is not the case. Rather, in this limit, one obtains for a particle of any shape in a duct of any cross section (B17)

$$\Delta P^+ A = \frac{u_R}{V_m}\mathbf{F} \tag{143}$$

Here, u is the local fluid velocity which would ensue if fluid flowed through the duct at mean velocity V_m in the absence of the particle. For example, in a circular duct the Poiseuille velocity distribution gives $u_R/V_m = 2(1-\beta^2)$, where β is the fractional distance of the center of reaction of the particle from the cylinder axis. Also, if the particle is spherical, $\mathbf{F} = -6\pi\mu a\mathbf{U}_R$, providing we neglect conventional wall-effects. For a particle at the centerline of a symmetrical duct, the coefficient u_R/V_m in Eq. (143) is 2 for a circular duct, 20/9 for an equilateral triangular duct, and 2.093 for a square duct, these values being based on the known velocity distributions for one-dimensional flow through conduits of these cross sections (B4, pp. 67*ff.*) In none of these cases is the coefficient unity, as would normally be expected for an "unbounded" fluid.

The origin of the paradoxical result (143) can be seen by observing that, since the sum of all external forces acting on the fluid in the duct is zero, there must exist a shearing force $\mathbf{F}_w$ exerted by the fluid on the duct walls, such that (B17)

$$\mathbf{F}_w = \left(\frac{u_R}{V_m} - 1\right)\mathbf{F} \tag{144}$$

The existence of this force derives from the fact that fluid is dragged along by the settling particle. In order to maintain the condition of no net flow through any horizontal plane, a compensating reverse flow must occur near the walls. Since the fluid adheres to the duct walls this flow produces the shearing force $\mathbf{F}_w$. These facts are not surprising. What is surprising, however, is the fact that the force does not tend to zero as $a/l \to 0$. For in this limit the fluid in the proximity of the walls, being infinitely distant from the disturbance, must be *at rest*. And if the fluid is at rest, how can it produce a shearing force? The resolution of the paradox lies in the fact that even though the shearing motion and, hence, the stresses near the wall are only infinitesimal, they act over an area which is infinitely large in comparison with that of the particle. These infinitesimal stresses acting over an infinite area are thereby able to produce a finite force.

It is clear from the preceding discussion that the apparent paradox is intimately bound up with the no-slip condition at the duct walls. Were the fluid free to glide effortlessly over the walls, the homogeneous flow $\mathbf{u}$ would simply be a plug flow, for which $u/V_m = 1$ everywhere. In this event Eqs. (143) and (144) would adopt their originally anticipated forms.

Since u/V_m has its maximum value (>1) at the duct centerline, decreasing to zero at the walls, the force on the wall changes in both magnitude and *direction* as the radial position of the particle is altered. The force is zero at a fractional eccentricity of $\beta = 0.707$ in the case of a circular cylinder.

Equations (143) and (144) apply also to the important case where the fluid itself may be in a state of net motion relative to the walls (B17), e.g., in a Poiseuillian flow through a tube containing a suspended particle. Here, however, ΔP^+ and $\mathbf{F}_w$ must be interpreted as the *additional* pressure drop and force, above and beyond that which would be observed in the absence of the particle with fluid flowing at the same mean velocity $\mathbf{V}_m$. Furthermore, $\mathbf{F}$ is the force on the particle due to both its own motion and that of the fluid, e.g., $\mathbf{F} = -6\pi\mu a(\mathbf{U}_R - \mathbf{u}_R)$ for a spherical particle in the absence of conventional wall-effects.

Equation (143) and, hence, indirectly (144) have been experimentally confirmed for both a settling particle in a stationary fluid (P9) and a stationary particle in a moving fluid (B17), the latter using the data of Fayon (F3).

Equation (143) may be applied to dilute multiparticle systems by simply summing the pressure drops for each of the separate particles (B12, H8). When a large body is immersed in a suspension of much smaller particles, customary practice is to calculate the "buoyant force" on the larger body by employing Archimedes' law in conjunction with the *mean density*, ρ_m, of the suspension. This assumes, in effect, that a pressure gradient exists within the "homogeneous" suspension which may be calculated by the hydrostatic formula

$$\nabla p = -\rho_m \mathbf{g} \tag{145}$$

where $\mathbf{g}$ is the local acceleration of gravity vector, directed vertically downward. This static viewpoint is obviously erroneous, since the particles are continuously settling—even if at imperceptibly small rates in the case of very fine particles or very viscous fluids. Nevertheless, one could prove that the result is valid were it permissible to assume that the fluid is laterally unbounded. For in this case there is no possibility of wall forces; whence the pressure-drop force must balance the drag forces on the particles. But according to Eqs. (143) and (144) this assumption is unwarranted, and so the usual "proof" of the validity of Archimedes' law for suspensions cannot be correct. Rather fortuitously, however, we find upon integrating (143) over the duct cross section that for *homogeneous* suspensions (B12)

$$A \sum_i \Delta P_i^+ = \sum_i \mathbf{F}_i$$

(i = ith particle). This relation depends entirely on the fact that, by definition,

$$\frac{1}{AV_m} \int_A u \, dA = 1$$

where dA is an element of cross sectional duct area. Now

$$\mathbf{F}_i + (\rho_p - \rho) v_i \mathbf{g} = \mathbf{0}$$

where ρ_p is the particle density, ρ the homogeneous fluid density, and v_i the volume of the ith particle. If L is the length of duct containing the suspended particles and $\Delta P^+ = \Sigma_i \, \Delta P_i^+$ is the total additional pressure drop, then

$$\Delta P^+ / L = -(\rho_p - \rho) \phi \mathbf{g}$$

where $\phi = \Sigma_i \, v_i / AL$ is the fractional concentration of solid particles. Since, by the definition of mean density, $(\rho_p - \rho)\phi = \rho_m - \rho$, we find that the additional pressure gradient, $\nabla p^+ = \Delta P^+ / L$, due to the presence of the particles is

$$\nabla p^+ = -(\rho_m - \rho) \mathbf{g}$$

This is the pressure gradient above and beyond that due to the homogeneous fluid alone, the latter being

$$\nabla p^{\bullet} = -\rho \mathbf{g}$$

The total pressure is $p = p^{\bullet} + p^{+}$. Upon adding the last two equations one arrives at Eq. (145).

Archimedes' law is thus, in fact, valid for suspensions—but not for the reasons usually stated. Similar remarks apply to the usual "proof" that the pressure drop in a fluidized bed is equal to the net weight of the bed divided by the cross sectional area (B12, H8). Based on this analysis it seems likely that dilute, laterally inhomogeneous suspensions would display deviations from Archimedes' law.

The pressure drop formula given by Eq. (143) can be explicitly corrected for conventional wall-effects as follows. By a slight generalization of Eqs. (31)–(34) of Brenner (B17), one has for a spherical particle of radius a that[18]

$$\Delta P^{+} \cdot \mathbf{V}_m A = \left[\mathbf{u}_R + \frac{a^2}{6\mu}(\nabla p^0)_R\right] \cdot \mathbf{F} + \boldsymbol{\omega}_R{}^f \cdot \mathbf{T}_R + \frac{5}{2} v(2\mu \mathbf{S}_R : \mathbf{S}_R) + O(a/l)^3 \tag{146}$$

where p^0 is the pressure arising from the unperturbed, unidirectional flow $\mathbf{u}$ through the duct; $\boldsymbol{\omega}^f = \frac{1}{2}(\nabla \times \mathbf{u})$ is the angular velocity of a fluid particle; $v = 4\pi a^3/3$ is the volume of the sphere; $\mathbf{S} = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\dagger}]$ is the rate of strain dyadic. As usual, the subscript R refers to evaluation at the center of the space presently occupied by the sphere. The force and torque on the sphere are, respectively, of the forms

$$\mathbf{F} = -6\pi\mu a \left[\frac{\mathbf{U}_R - \mathbf{u}_R - (a^2/6\mu)(\nabla p^0)_R}{1 - \underline{W}_R(a/l)} + O(a/l)^3\right]$$

$$\mathbf{T}_R = -8\pi\mu a^3(\boldsymbol{\omega} - \boldsymbol{\omega}_R{}^f) + O(a/l)^3$$

where $\underline{W}_R$ is the nondimensional, wall-effect factor for motion of the sphere parallel to the axis of the cylinder; e.g., $\underline{W}_R = f(\beta)$ for a circular cylinder. In

[18] Only the term ∇p^0 does not appear in the original reference (B17). It arises by including second derivatives in the Taylor series expansion of the unperturbed Stokes flow about R,

$$\mathbf{u} = \mathbf{u}_R + \boldsymbol{\omega}_R{}^f \times \mathbf{r} + \mathbf{S}_R \cdot \mathbf{r} + \mathbf{rr} : (\nabla\nabla \mathbf{u})_R + \cdots$$

by observing that $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{I} : \nabla\nabla \mathbf{u} = \nabla^2 \mathbf{u} = \mu^{-1} \nabla p^0$. Note the close relationship to Faxén's law, Eq. (89).

the case of a circular cylinder of radius l we have

$$\begin{aligned}\mathbf{u}_R &= 2\mathbf{V}_m(1-\beta^2)\\ \nabla p^0 &= -8\mu\mathbf{V}_m/l^2\\ \boldsymbol{\omega}_R{}^f &= 2\mathbf{V}_m \times \mathbf{b}/l^2\\ \mathbf{S}_R:\mathbf{S}_R &= 8\mathbf{V}_m\cdot\mathbf{V}_m b^2/l^4\end{aligned}$$

where $\mathbf{b}$ is the position vector of the sphere center relative to the tube axis. These make

$$\Delta P^+(\pi l^2) = 2\left[(1-\beta^2) - \frac{2}{3}\left(\frac{a}{l}\right)^2\right]\mathbf{F} + \frac{2\mathbf{b}}{l^2}\times\mathbf{T}_R + \frac{160}{3}\pi\mu a^3\beta^2\mathbf{V}_m + O\left(\frac{a}{l}\right)^3 \tag{147a}$$

in which $\mathbf{F}$ is given by Eq. (142), and

$$\mathbf{T}_R = -8\pi\mu a^3\left(\boldsymbol{\omega} - 2\mathbf{V}_m\times\frac{\mathbf{b}}{l^2}\right) + O\left(\frac{a}{l}\right)^3$$

This expression for the pressure drop agrees in all particulars with a comparable formula obtained via a detailed solution of Stokes equations for the motion of a sphere in a circular cylinder (B28; unpublished calculations).

The couple on the sphere vanishes unless it is restrained from rotating. If the sphere is also neutrally buoyant then $\mathbf{F} = \mathbf{0}$, and only the last term in Eq. (147a) survives. By noting that the local rate of mechanical energy dissipation in the unperturbed flow is $2\mu\mathbf{S}_R:\mathbf{S}_R$, this ultimately leads to a simple proof of Einstein's law of suspension viscosity (E1a) for flow through cylinders (B17), provided that the spheres are randomly distributed over the duct cross section.

For a sphere settling under the influence of gravity in a quiescent fluid ($\mathbf{V}_m = \mathbf{0}$), Eq. (147a) reduces to

$$\Delta P^+ A = 2\left[(1-\beta^2) - \frac{2}{3}\left(\frac{a}{l}\right)^2\right]\mathbf{F} + O\left(\frac{a}{l}\right)^3 \tag{147b}$$

where

$$\mathbf{F} = -\frac{6\pi\mu a\mathbf{U}_R}{1 - f(\beta)(a/l)} + O\left(\frac{a}{l}\right)^3$$

This result [or more generally, Eq. (147a)] constitutes the extension of Eq. (143) to situations where conventional wall effects are not negligible. It is interesting to note that Archimedes' law is now no longer strictly valid; rather, in place of Eq. (145) we now obtain

$$\nabla p = -\rho_m\mathbf{g}\left[1 - \frac{4}{3}\left(\frac{a}{l}\right)^2\left(1 - \frac{\rho}{\rho_m}\right) + O\left(\frac{a}{l}\right)^3\right]$$

for a dilute, homogeneous suspension of equisized spheres in a circular cylinder. The comparable result for spherical particles in a cylinder of any cross section is obtained by replacing the factor $\frac{4}{3}(a/l)^2$ by $a^2|\nabla p^0|/6\mu V_m$. For example, from the known solution for laminar flow in an elliptical duct (B4, p. 69) of semiaxes b and c one obtains $2a^2(b^2 + c^2)/3b^2c^2$ as the appropriate wall-correction factor.

Closely related to the preceding is the problem of calculating the pressure drop due to Stokes flow through a cylinder of arbitrary (but constant) cross section for *arbitrary* boundary conditions on the surfaces bounding the cylinder. A simple application of the Reciprocal theorem (B18) permits one to express this pressure drop *directly* in terms of the prescribed velocity field on the cylinder walls, top, and bottom. If $(\mathbf{v}, \boldsymbol{\pi})$ and $(\mathbf{v}^{(0)}, \boldsymbol{\pi}^{(0)})$ denote the velocity and dyadic stress fields associated with any two solutions of Stokes equations, one may write

$$\oint_{S_w + S_t + S_b} d\mathbf{S} \cdot \boldsymbol{\pi} \cdot \mathbf{v}^{(0)} = \oint_{S_w + S_t + S_b} d\mathbf{S} \cdot \boldsymbol{\pi}^{(0)} \cdot \mathbf{v}$$

where S_w, S_t and S_b refer, respectively, to the cylinder walls, top, and bottom. Choose $\mathbf{v}^{(0)}$ to be the classical, no-slip, unidirectional laminar flow through the cylinder corresponding to a *unit* mean velocity of flow (e.g., for a circular cylinder, $\mathbf{v}^{(0)} = \mathbf{k}u$, where $u = 2[1 - (R/l)^2]$). Now $\mathbf{v}^{(0)}$ vanishes on S_w; furthermore, $d\mathbf{S} \cdot \boldsymbol{\pi} \cdot \mathbf{v}^{(0)} = \pm pudA$ on S_t and S_b, where p is necessarily constant for the class of problems to which the present method of calculation is applicable (otherwise the definition of pressure drop is ambiguous). Since $\int udA = A$ over both S_t and S_b, the left-hand integral of the preceding displayed equation is simply ΔPA. Hence,

$$\Delta P = \frac{1}{A} \oint_{S_w + S_t + S_b} d\mathbf{S} \cdot \boldsymbol{\pi}^{(0)} \cdot \mathbf{v}$$

Calculation of the pressure drop from the prescribed field $\mathbf{v}$ on S_w, S_t, and S_b is thus reduced to a quadrature whenever the classical solution for laminar flow through the cylinder is already known. A relationship comparable to the above was previously given for the special case of a circular cylinder by Brenner (B13) along with an example of its utility in applications.

D. Multiparticle Systems

1. *Finite Particle Systems*

a. Quiescent Fluids. In this section we consider situations where a *finite* number of solid particles, each of any shape, in any relative positions and orientations, move through a fluid. Initially, attention is confined to the case where the fluid is unbounded and at rest at infinity. Let O_i be any origin fixed, once and for all, in the ith particle. The hydrodynamic force $\mathbf{F}_i$ and torque $\mathbf{T}_i$ (about O_i) exerted by the fluid on the ith particle are then linear vector functions of the translational velocities $\mathbf{U}_j$ (origin O_j) and angular velocities $\boldsymbol{\omega}_j$ of *all* the particles. In particular, we have (B22)

$$\mathbf{F}_i = -\mu \sum_{j=1}^{n} \left[\overset{(t)}{\mathbf{K}}_{ij} \cdot \mathbf{U}_j + \overset{(c)}{\mathbf{K}}{}^{\dagger}_{ji} \cdot \boldsymbol{\omega}_j \right] \tag{148}$$

$$\mathbf{T}_i = -\mu \sum_{j=1}^{n} \left[\overset{(c)}{\mathbf{K}}_{ij} \cdot \mathbf{U}_j + \overset{(r)}{\mathbf{K}}_{ij} \cdot \boldsymbol{\omega}_j \right] \tag{149}$$

where $i = 1, 2, 3, \ldots, n$, the latter being the total number of particles in the system.

Each constant $\mathbf{K}_{ij}$ resistance dyadic is an intrinsic geometric property of the instantaneous configuration of the entire particle system, dependent upon the sizes, shapes, mode of arrangement, and relative orientations of all the particles. $\overset{(r)}{\mathbf{K}}_{ij}$ depends on the choice of O_i and O_j, whereas $\overset{(c)}{\mathbf{K}}_{ij}$ depends only on the choice of O_i. The translation dyadics, $\overset{(t)}{\mathbf{K}}_{ij}$, are origin-independent. The various resistance dyadics are independent of the fluid viscosity and of the velocities and spins of all the particles. It should be noted that the two indices associated with each dyadic refer not to tensor indices, but rather to numbers used in identifying the individual particles. For example, $\overset{(r)}{\mathbf{K}}_{ij}$ is associated with the torque on the ith particle due to the rotation of the jth particle while all other particles are at rest. These dyadics satisfy the symmetry relations (B22)

$$\overset{(t)}{\mathbf{K}}_{ij} = \overset{(t)}{\mathbf{K}}{}^{\dagger}_{ji} \tag{150a}$$

$$\overset{(r)}{\mathbf{K}}_{ij} = \overset{(r)}{\mathbf{K}}{}^{\dagger}_{ji} \tag{150b}$$

When the two indices of these "direct" dyadics are equal, the resulting dyadics are symmetric and positive-definite.

The dyadic resistance coefficients derive from the quasi-steady Stokes equations as follows: Let $(\mathbf{V}_j, \mathbf{P}_j)$ be the intrinsic solutions of the dyadic Stokes equations [cf. Eqs. (17) and (18)] associated with the motion of the jth particle while all other particles are at rest. Explicitly, let the translational solutions satisfy the dyadic boundary conditions

$$\overset{(t)}{\mathbf{V}}_j = \begin{cases} \mathbf{I} & \text{on } S_j \\ \mathbf{0} & \text{on } S_k \quad (k \neq j) \quad (k = 1, 2, 3, \ldots, n) \\ \mathbf{0} & \text{at infinity} \end{cases} \tag{151}$$

and let the rotational solutions satisfy

$$\overset{(r)}{\mathbf{V}}_j = \begin{cases} \boldsymbol{\varepsilon} \cdot \mathbf{r}_j & \text{on } S_j \\ \mathbf{0} & \text{on } S_k \quad (k \neq j) \quad (k = 1, 2, 3, \ldots, n) \\ \mathbf{0} & \text{at infinity} \end{cases} \tag{152}$$

where S_i denotes the surface of the ith particle, and $\mathbf{r}_i$ is measured from O_i. If the intrinsic triadic stress fields deriving from these solutions [cf. Eqs. (34) and (37)] are denoted by $\boldsymbol{\Pi}_j$, then the direct resistance coefficients may be calculated from the relations

$$\overset{(t)}{\mathbf{K}}_{ij} = -\int_{S_i} d\mathbf{S} \cdot \overset{(t)}{\boldsymbol{\Pi}}_j \tag{153}$$

$$\overset{(r)}{\mathbf{K}}_{ij} = -\int_{S_i} \mathbf{r}_i \times \left(d\mathbf{S} \cdot \overset{(r)}{\boldsymbol{\Pi}}_j\right) \tag{154}$$

while the cross-coefficients may be obtained either from the relation

$$\overset{(c)}{\mathbf{K}}_{ij} = -\int_{S_i} \mathbf{r}_i \times \left(d\mathbf{S} \cdot \overset{(t)}{\boldsymbol{\Pi}}_j\right) \tag{155a}$$

or from the equivalent relation

$$\overset{(c)}{\mathbf{K}}_{ij} = -\int_{S_j} d\mathbf{S} \cdot \overset{(r)}{\boldsymbol{\Pi}}_i{}^{\dagger} \tag{155b}$$

The unity of this mode of representing the intrinsic resistance of finite multiparticle systems is best shown by resorting to a matrix representation of Eqs. (148) and (149). Define the four, partitioned $3n \times 1$ column matrices $\|F\|$, $\|T\|$, $\|U\|$, $\|\omega\|$ by expressions of the general form

$$\|F\| = \left\| \begin{matrix} \|\mathbf{F}_1\| \\ \|\mathbf{F}_2\| \\ \vdots \\ \|\mathbf{F}_n\| \end{matrix} \right\|, \qquad \text{etc.} \tag{156}$$

in which

$$\|\mathbf{F}_i\| = \left\| \begin{matrix} F_i^{(1)} \\ F_i^{(2)} \\ F_i^{(3)} \end{matrix} \right\| \tag{157}$$

is a 3×1 column matrix whose three scalar elements, $F_i^{(1)}$, $F_i^{(2)}$, $F_i^{(3)}$, are the components of the vector force $\mathbf{F}_i$ in any convenient Cartesian system.

Also define the $3n \times 3n$ square, partitioned matrices $\|\overset{(t)}{K}\|$, $\|\overset{(r)}{K}\|$, $\|\overset{(c)}{K}\|$ by expressions of the general form

$$\left\|\overset{(t)}{K}\right\| = \left\| \begin{matrix} \left\|\overset{(t)}{\mathbf{K}}_{11}\right\| & \left\|\overset{(t)}{\mathbf{K}}_{12}\right\| \cdots \left\|\overset{(t)}{\mathbf{K}}_{1n}\right\| \\ \left\|\overset{(t)}{\mathbf{K}}_{21}\right\| & \left\|\overset{(t)}{\mathbf{K}}_{22}\right\| \cdots \left\|\overset{(t)}{\mathbf{K}}_{2n}\right\| \\ \vdots & \vdots \quad \vdots \quad \vdots \\ \left\|\overset{(t)}{\mathbf{K}}_{n1}\right\| & \left\|\overset{(t)}{\mathbf{K}}_{n2}\right\| \cdots \left\|\overset{(t)}{\mathbf{K}}_{nn}\right\| \end{matrix} \right\|, \qquad \text{etc.} \tag{158}$$

in which

$$\left\|\overset{(t)}{\mathbf{K}}_{ij}\right\| = \left\| \begin{matrix} \overset{(t)}{K}{}_{ij}^{(11)} & \overset{(t)}{K}{}_{ij}^{(12)} & \overset{(t)}{K}{}_{ij}^{(13)} \\ \overset{(t)}{K}{}_{ij}^{(21)} & \overset{(t)}{K}{}_{ij}^{(22)} & \overset{(t)}{K}{}_{ij}^{(23)} \\ \overset{(t)}{K}{}_{ij}^{(31)} & \overset{(t)}{K}{}_{ij}^{(32)} & \overset{(t)}{K}{}_{ij}^{(33)} \end{matrix} \right\| \tag{159}$$

is a 3×3 matrix whose nine scalar elements are the components of the dyadic $\overset{(t)}{\mathbf{K}}_{ij}$ in the same Cartesian system as used for the column matrices.

If we now define the $6n \times 1$ partitioned column matrices

$$\|\mathscr{F}\| = \left\| \begin{matrix} \|F\| \\ \|T\| \end{matrix} \right\|, \qquad \|\mathscr{U}\| = \left\| \begin{matrix} \|U\| \\ \|\omega\| \end{matrix} \right\| \tag{160}$$

and the $6n \times 6n$ square, partitioned matrix

$$\|\mathscr{K}\| = \left\| \begin{matrix} \left\|\overset{(t)}{K}\right\| & \left\|\overset{(c)}{K}\right\|^{\dagger} \\ \left\|\overset{(c)}{K}\right\| & \left\|\overset{(r)}{K}\right\| \end{matrix} \right\| \tag{161}$$

then Eqs. (148) and (149) are equivalent to the single matrix equation (B22)

$$\|\mathscr{F}\| = -\mu\|\mathscr{K}\| \, \|\mathscr{U}\| \tag{162}$$

[cf. Eq. (49) for a *single* particle.]

Incorporated into the $\|\mathscr{K}\|$ matrix is the intrinsic resistance of the entire particle system. It is called the *grand resistance matrix*. In view of the symmetry relations

$$\left\|\overset{(t)}{K}\right\| = \left\|\overset{(t)}{K}\right\|^{\dagger}, \qquad \left\|\overset{(r)}{K}\right\| = \left\|\overset{(r)}{K}\right\|^{\dagger} \tag{163}$$

the grand resistance matrix is symmetric:

$$\|\mathscr{K}\| = \|\mathscr{K}\|^{\dagger} \tag{164}$$

Furthermore, the latter is positive-definite, as are the direct submatrices $\left\|\overset{(t)}{K}\right\|$ and $\left\|\overset{(r)}{K}\right\|$ too. The scalar elements of these matrices must therefore satisfy a host of inequalities [see the paragraph containing Eq. (50)].

Insight into the internal structure of the multiparticle resistance dyadics may be obtained from the two-sphere example discussed by Brenner (B22). Let a_1 and a_2 be the radii of the spheres; $\mathbf{e}_{12}$ is a unit vector drawn from the center O_1 of sphere 1 to the center O_2 of sphere 2, and $2h$ is the center-to-center distance. The resistance dyadics are then, to terms of the lowest orders in a_i/h,

$$\overset{(t)}{\mathbf{K}}_{11} = 6\pi a_1\left[\mathbf{I} + \frac{9}{64}\left(\frac{a_1}{h}\right)\left(\frac{a_2}{h}\right)(\mathbf{I} + 3\mathbf{e}_{12}\mathbf{e}_{12}) + \mathrm{o}\left(\frac{a_i}{h}\right)^2\right] \tag{165a}$$

$$\overset{(r)}{\mathbf{K}}_{11} = 8\pi a_1{}^3\left[\mathbf{I} + \frac{3}{64}\left(\frac{a_2}{h}\right)\left(\frac{a_1}{h}\right)^3(\mathbf{I} - 3\mathbf{e}_{12}\mathbf{e}_{12}) + \mathrm{o}\left(\frac{a_i}{h}\right)^4\right] \tag{165b}$$

$$\overset{(c)}{\mathbf{K}}_{11} = -\frac{9}{16}\pi a_1 a_2\left[\left(\frac{a_1}{h}\right)^3 \boldsymbol{\varepsilon}\cdot\mathbf{e}_{12} + \mathrm{o}\left(\frac{a_i}{h}\right)^3\right] \tag{165c}$$

$$\overset{(t)}{\mathbf{K}}_{12} = -\frac{9}{4}\pi a_1\left[\left(\frac{a_2}{h}\right)(\mathbf{I} + \mathbf{e}_{12}\mathbf{e}_{12}) + \mathrm{o}\left(\frac{a_i}{h}\right)\right] \tag{165d}$$

$$\overset{(r)}{\mathbf{K}}_{12} = \frac{1}{2}\pi a_1{}^3\left[\left(\frac{a_2}{h}\right)^3(\mathbf{I} - 3\mathbf{e}_{12}\mathbf{e}_{12}) + \mathrm{o}\left(\frac{a_i}{h}\right)^3\right] \tag{165e}$$

$$\overset{(c)}{\mathbf{K}}_{12} = \frac{3}{2}\pi a_1 a_2\left[\left(\frac{a_1}{h}\right)^2 \boldsymbol{\varepsilon}\cdot\mathbf{e}_{12} + \mathrm{o}\left(\frac{a_i}{h}\right)^2\right] \tag{165f}$$

The $\mathbf{K}_{22}$ and $\mathbf{K}_{21}$ coefficients may be obtained from these by interchanging the indices, bearing in mind that $\mathbf{e}_{21} = -\mathbf{e}_{12}$. Note that Eqs. (165) correctly satisfy the symmetry relations cited in Eq. (150).

A much more complete discussion of recent work on the interaction of two and more spherical particles, including a detailed comparison with experimental data, will be found in Happel and Brenner (H9). To the list of references given there should be added the experimental work of Bammi [(B1); summarized by McNown (M8)], Timbrell (T5), Andersson (A3), Eveson (E3), Gasparian (G2), and Jayaweera *et al.* (J2). Andersson (A4) presents data on the sedimentation of two rods. In addition there are recent theoretical analyses by Kaufman (K6) on the axisymmetric motion of two equal spheres, and Hocking (H14) on the motion of clusters of identical spheres. Pshenai-Severin (P12, P13) and Hocking (H13a) theoretically treat the case where the spheres are of unequal size. Berker (B4, p. 247) presents some interesting general theorems pertaining to the motion of two equal spheres. More recently, Goldman, Cox, and Brenner (G5b), using bipolar coordinates, solved *exactly* the problem of two identical spheres for the general case where their line of centers is arbitrarily inclined relative to the direction of gravity. Their treatment includes the case where the spheres fall without rotation (i.e., like a rigid dumbbell), as well as the case where the individual spheres rotate freely about their own centers in response to the hydrodynamic couple caused by the translational motion. Their two-sphere results, which apply for all values of a/h—including the limiting case where the spheres touch, supercede the approximate results discussed in Chapter 6 of Happel and Brenner (H9).

With regard to systems consisting of more than two spherical particles settling in an unbounded, quiescent fluid, Srimathi and Bhat (S17) present experimental data on drag interaction effects when a number of identical spheres rigidly joined together in a line by a thin rod are allowed to settle.

The multiple-sphere experiments of Jayaweera *et al.* (J2) point up a truly remarkable ordering phenomenon. Three to six equal spheres released as a compact cluster in a viscous fluid eventually arrange themselves in the same horizontal plane at the vertices of the corresponding regular polygon, provided that the Reynolds number is sufficiently small ($R = 0.06$ to 7.0 in the experiments, these being based on sphere radius and settling velocity). The polygon dimensions continuously expand during its fall, but at an ever-decreasing rate. If the number of spheres exceeds six, or if $R > 7$, the cluster exhibits no tendency to form a regular polygon, but rather separates into two or more groups. On the basis of Stokes equations, Hocking (H14) furnished a partial theoretical analysis of these phenomena by investigating the stability of a configuration of n equal spheres at the vertices of a regular horizontal polygon. The distance between adjacent spheres was assumed sufficiently large compared with the sphere radii to permit the use of only first-order hydrodynamic interactions. Hocking showed that the configuration was neutrally stable to small perturbations for $n = 3$ to 6, but unstable for $n \geqslant 7$—thus explaining why no regular polygonal array was observed for more than six spheres. However, he was unable to explain the increasing size of the polygon, or to prove positive stability for $n = 3$ to 6 on the basis of the Stokes equations; rather, the small oscillations of the spheres were of constant amplitude. Bretherton (B31) completed the analysis by allowing for the existence of small, nonzero inertial effects in the equations of motion. Thereby he demonstrated, among other things, that the observed increase in the size of the array and the damping of the oscillations for $3 \leqslant n \leqslant 6$ were due to the inertia of the fluid.

First-order interactions between two particles of any shape translating and rotating in an unbounded fluid at rest at infinity can be directly expressed in terms of the fundamental resistance dyadics of the individual particles themselves. Denote by $\overset{(\ )}{\mathbf{K}}_1$ and $\overset{(\ )}{\mathbf{K}}_2$ the appropriate resistance dyadics for particles 1 and 2, respectively, when each is alone in the unbounded fluid. The origins O_1 and O_2 to which the rotation and coupling dyadics are referred may be any points fixed in each of the particles. Let **e** be a unit vector drawn along the line from O_1 and O_2 or vice versa; since **e** always occurs as the dyad **ee** it is immaterial which of the two possible directions is selected. The multiparticle resistance dyadics are then expressible in the form (B20, B22, C20)

$$\overset{(t)}{\mathbf{K}}_{11} = \overset{(t)}{\mathbf{K}}_1 + (16\pi h)^{-2}\overset{(t)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_1 + \mathrm{O}(a/h)^3 \quad (166a)$$

$$\overset{(r)}{\mathbf{K}}_{11} = \overset{(r)}{\mathbf{K}}_1 + (16\pi h)^{-2}\overset{(c)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(c)}{\mathbf{K}}_1{}^{\dagger} + \mathrm{O}(a/h)^3 \quad (166b)$$

$$\overset{(c)}{\mathbf{K}}_{11} = \overset{(c)}{\mathbf{K}}_1 + (16\pi h)^{-2}\overset{(c)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_1 + \mathrm{O}(a/h)^3 \quad (166c)$$

$$\overset{(t)}{\mathbf{K}}_{12} = -(16\pi h)^{-1}\overset{(t)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 + \mathrm{O}(a/h)^2 \quad (166d)$$

$$\overset{(r)}{\mathbf{K}}_{12} = -(16\pi h)^{-1}\overset{(c)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(c)}{\mathbf{K}}_2{}^{\dagger} + \mathrm{O}(a/h)^2 \quad (166e)$$

$$\overset{(c)}{\mathbf{K}}_{12} = -(16\pi h)^{-1}\overset{(c)}{\mathbf{K}}_1 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 + \mathrm{O}(a/h)^2 \quad (166f)$$

where a is a characteristic particle dimension and $2h$ is the "center-to-center" distance $|\overrightarrow{O_1 O_2}|$. The remaining dyadics, $\overset{(\ )}{\mathbf{K}}_{22}$ and $\overset{(\ )}{\mathbf{K}}_{21}$, may be obtained by permuting the indices. Observe that these coefficients correctly satisfy the symmetry conditions set forth in Eq. (150). Under relatively mild symmetry conditions imposed upon the shapes of the two particles, Eqs. (166a) and (166d) can be expressed in the considerably more accurate forms (C20)

$$\overset{(t)}{\mathbf{K}}_{11} = \overset{(t)}{\mathbf{K}}_1 \cdot \left[\mathbf{I} - (16\pi h)^{-2}(\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_1\right]^{-1} + \mathrm{O}(a/h)^3$$

$$\overset{(t)}{\mathbf{K}}_{12} = -(16\pi h)^{-1}\overset{(t)}{\mathbf{K}}_{11} \cdot (\mathbf{I} + \mathbf{ee}) \cdot \overset{(t)}{\mathbf{K}}_2 + \mathrm{O}(a/h)^3$$

with comparable expressions for $\overset{(t)}{\mathbf{K}}_{22}$ and $\overset{(t)}{\mathbf{K}}_{21}$ obtained by permuting the indices.

Equations of the form (148)–(149), as well as their matrix counterparts, continue to be applicable even if the fluid is partly or wholly bounded by nondeformable boundaries on which no work can be done by the stresses, i.e., solid boundaries and nondeformable free surfaces. The only modification required in the analysis is that the boundary conditions (151) and (152) must be supplemented by the additional boundary conditions appropriate to these external boundaries—for example, $\overset{(t)}{\mathbf{V}}_j = \mathbf{0}$ and $\overset{(r)}{\mathbf{V}}_j = \mathbf{0}$ on S_w if the surface S_w of the boundary is solid.

b. Shear and Higher-Order Flows. If the fluid is in a state of net Stokes flow with velocity $\mathbf{u} = \mathbf{u}(\mathbf{r})$ at infinity, the force and torque (about O_i) on the ith particle in a multiparticle system are given by (B26)

$$\mathbf{F}_i = -\mu \sum_{j=1}^{n} \mathfrak{F}_{ij} \cdot (\mathbf{U}_j + \boldsymbol{\omega}_j \times \mathbf{r}_j - \mathbf{u}) \tag{167}$$

$$\mathbf{T}_i = -\mu \sum_{j=1}^{n} \mathfrak{T}_{ij} \cdot (\mathbf{U}_j + \boldsymbol{\omega}_j \times \mathbf{r}_j - \mathbf{u}) \tag{168}$$

The dyadics $\mathfrak{F}_{ij}$ and $\mathfrak{T}_{ij}$ are two-index, symbolic operators termed the multiparticle force and torque operators. They are defined by the relations

$$\mathfrak{F}_{ij} = \int_{S_j} d\mathbf{S} \cdot \overset{(t)}{\mathbf{\Pi}}{}_i^{\dagger} \exp(\mathbf{r}_j \cdot \nabla_j) \tag{169}$$

and

$$\mathfrak{T}_{ij} = \int_{S_j} d\mathbf{S} \cdot \overset{(r)}{\mathbf{\Pi}}{}_i^{\dagger} \exp(\mathbf{r}_j \cdot \nabla_j) \tag{170}$$

These operators are intrinsic properties of the geometrical configuration of the multiparticle system.

Equations (167) and (168) express the fact that the force and torque on each particle are linear vector functionals of the "slip velocities" of *all* the particles. It should be clearly noted that these operators are calculable solely from a knowledge of the intrinsic translational and rotational solutions of Stokes equations resulting from motion of the jth particle when *all the other particles, as well as the fluid at infinity, are at rest.* In the case where $\mathbf{u} = \mathbf{0}$, Eqs. (167) and (168) reduce to (148) and (149), respectively. As yet, there exists no system for which these operators are known.

The interaction of two rigid spherical particles in a shear field have been experimentally investigated by Mason and co-workers. Their results are summarized by Mason and Bartok (M5) and by Goldsmith and Mason (G9d), where references to earlier work will be found.

c. Wall-Effects. Equations of the form (167) and (168) continue to be applicable even when the fluid is partially bounded externally by rigid walls or nondeformable free surfaces on which the stresses do no work. Equations (151) and (152), from which the intrinsic stress triadics in (169) and (170) derive, must, of course, be supplemented by the appropriate boundary conditions at the bounding surfaces. With this further generalization, Eqs. (167) and (168) summarize completely *all* the prior general force and torque formulas.

By superposing solutions for the Stokes flow resulting from the action of isolated point forces directed along the axis of a circular cylinder, Sonshine and Brenner (S15) have computed the wall-effect on the motion of a number of identical, equally-spaced particles settling in a single column along the axis of an infinitely long circular cylinder. With a a characteristic particle dimension, $2h$ the distance between adjacent particles, and l the cylinder radius, the analysis requires $a/h \ll 1$ and $a/l \ll 1$, but applies for all values of h/l in the range $0 \leqslant h/l < \infty$. Calculations are presented for $n = 1, 2, 3, \ldots, \infty$ particles. When n is finite, the drag force on each of the particles remains finite as the wall recedes to infinity ($h/l \to 0$ and $a/l \to 0$). [See also the experimental data of Srimathi and Bhat (S17).] On the other hand, as the number of particles is increased indefinitely, the drag force on each particle becomes infinite when the walls are removed, despite the fact that it remains finite when walls are present. This divergence in unbounded media accords with previous calculations of Burgers (B38) on the related problem of three-dimensional periodic arrays (H10). The contrasting behavior of the bounded vs unbounded cases stems from the fact that the motion generated by the action of a single point force decays exponentially with axial distance in the former case, but only inversely with distance in the latter case (see also footnote 48). The fundamental role played by an encircling boundary in determining the settling velocity of a dilute suspension of particles has already been recognized by Famularo and Happel (F1). It appears that their analysis could be greatly simplified by applying the point force summation techniques of Sonshine and Brenner (S15) to the case where the particles are free to occupy any position within the circular cylinder (B28).

2. *Infinite Particle Systems*

The topic of flow through an effectively infinite system of particles belongs to the more general domain of flow through porous media (C12, C13, P10, R5, S3, S4). In the absence of physicochemical interaction between the particles and fluid, the slow quasi-static Newtonian flow of incompressible fluids through such media is governed by Darcy's law. This linear phenomenological law has the form

$$\mathbf{q} = -(\mathbf{k}/\mu) \cdot \nabla \not{p} \tag{171}$$

to which is appended the condition of incompressibility,

$$\nabla \cdot \mathbf{q} = 0 \tag{172}$$

$\mathbf{q}$ is the local volumetric flow rate per unit of superficial cross sectional area (*filter velocity*) at a "point" in the medium, $\not{p}$ is the average local dynamic pressure, and $\mathbf{k}$ is the local permeability dyadic. The latter is an intrinsic geometric property of the porous medium itself. For isotropic media, $\mathbf{k} = \mathbf{I}k$, where k is a scalar; hence,

$$\mathbf{q} = -(k/\mu)\nabla \not{p} \tag{173}$$

Darcy's law is an empirical relation derived by induction from experimental data. Though many attempts have been made to "derive" this law directly from the Navier–Stokes equations—especially from the linearized form of the latter—the legitimacy and generality of these "proofs" remain open questions (S4, p. 661). A primary aim of these treatments is to establish the relationship between the permeability coefficient and the geometrical structure of the porous medium. An even broader goal is the elucidation of the structure of the "subcontinuum" velocity field $\mathbf{v}$ in the proximity of the individual particles, from which $\mathbf{q}$ ultimately derives by appropriate averaging. Only through such more detailed knowledge can one calculate from first principles such quantities as heat- and mass-transfer coefficients, and the degree of macroscopic dispersion ("mixing") in such media. In this connection, simple cell models of the types proposed by Happel (H4, H5, H7) and Kuwabara (K12) display significant engineering promise (H6, H9, P7, P8, R8).

Hasimoto's (H10) treatment of flow through *periodic* arrays of identical particles offers a variety of opportunities towards achieving these objectives. When the particles are arrayed in an infinite periodic lattice, the subcontinuum flow field itself becomes periodic, at least in so far as the Stokes approximation is concerned. The filter velocity and *macroscopic* pressure gradient, however, remain constant—as had previously been pointed out (B11, U1) in prior, less potent, attempts to treat the periodic flow problem. [The Stokes flow problem for flow through a cubic periodic array of touching spheres has recently been solved by Snyder and Stewart (S14) using a Galerkin approximation scheme. Their results furnish the detailed *local* velocity and pressure fields as well as the drag on the particles. Their calculated friction factors agree to within 5 per cent with the experimental values for the two configurations studied by them.] Thus, the concept of cell models with fluid boundaries is already implicit in the concept of periodic assemblages—but in a much less arbitrary, albeit complex, fashion than in Happel's treatment. In principle, Hasimoto's technique permits one to relate the gross properties of the porous medium directly to the geometric structure and arrangement of the individual particles comprising it. Though the original analysis (H10) of three-dimensional systems is limited to spherical particles, we have in mind its extension to anisotropic media consisting, say, of identical anisotropic particles, similarly oriented, in periodic array. It is already intuitively obvious, for example, that a periodic assemblage consisting of finite circular cylinders with axes parallel and with centers at the nodal points of a simple cubic lattice gives rise to an anisotropic porous medium in which the permeability dyadic is *symmetric*. The principal axes of $\mathbf{k}$ clearly lie parallel to those of the individual cylinders. In a more general context, Ripps and Brenner (R5b) have shown that *any* infinite collection of identical, similarly oriented particles in periodic array gives rise to a symmetric permeability tensor—irrespective of the shape of the particles comprising the array. The symmetry of this tensor is thus a consequence of the periodicity of the array alone, and does not further require that the particles themselves by symmetric. Moreover, it is shown (R5b) that if the individual particles possess screw symmetry the fluid will exert a torque on each particle. Such periodic models, coupled with the broader ideas of Dahler and Scriven (D2) on the general theory of *structured continua*, offer promise in advancing our understanding of flow through porous media.

In connection with the relationship between the macro- and microstructure of porous media, we wish to put forward some frankly speculative thoughts on what appears to be a fundamental limitation on the generality of Darcy's law in its presently accepted form. Consider two artificial porous media composed of identical particles differing only in their *screw-sense*; that is, suppose one medium is entirely composed of right-handed particles, and the other of the same species of left-handed particles. Since $\mathbf{q}$, $\mathbf{k}$, and $\not{p}$ are usually regarded as *true* tensors, Eqs. (171)-(172) predict that flows through these two media would be macroscopically indistinguishable. Only through the intervention of *pseudo*-tensors can a sense-difference be manifested. In view of our findings for isolated particles, and of our knowledge of other sense-dependent physical phenomena, e.g., magnetism, optical rotation, and the like, it seems unlikely that the screw-sense of the individual particles is somehow lost in the averaging process, as implied by Darcy's law. Rather, it is more likely that the law is incomplete—a defect we shall attempt to remedy. In what follows we follow in broad outline the general theory of structured continua advanced by Dahler and Scriven (D2).

The most likely manifestation of this screw-sense lies in the existence of a torque exerted by the fluid on the particles comprising the porous medium and vice versa. The vector $-\nabla\not{p}\, dV$ gives the force exerted by the surroundings on the contents of a small volume element dV. Hence, the external moment of the forces acting on a volume V of the medium, about some arbitray origin, is

$$\mathbf{M}_{\text{ext}} = -\int_V \mathbf{r} \times \nabla\not{p}\, dV$$

We now assume the existence of a local torque density pseudovector $\mathbf{t}$ such that $\mathbf{t}\, dV$ gives the torque exerted on the contents of dV by its immediate surroundings. There is thus an internal moment, which is independent of the choice of origin, such that

$$\mathbf{M}_{\text{int}} = \int_V \mathbf{t}\, dV$$

The total first moment about the origin is, therefore,

$$\mathbf{M} = -\int_V \mathbf{r} \times \nabla p \, dV + \int_V \mathbf{t} \, dV$$

The internal moment may be replaced by an *equivalent* macroscopic distribution by utilizing the identity

$$\mathbf{t} = \tfrac{1}{2}\mathbf{r} \times (\nabla \times \mathbf{t}) + \tfrac{1}{2}\nabla \cdot [(\mathbf{I} \times \mathbf{t}) \times \mathbf{r}]$$

whence, by the divergence theorem,

$$\int_V \mathbf{t} \, dV = \tfrac{1}{2}\int_V \mathbf{r} \times (\nabla \times \mathbf{t}) \, dV + \tfrac{1}{2}\oint_S (d\mathbf{S} \times \mathbf{t}) \times \mathbf{r}$$

where S is the closed surface bounding V.

Following Landau and Lifshitz (L7, pp. 37 and 114) and Dahler and Scriven (D2) we choose S *outside* of the isolated volume V of the body. Since $\mathbf{t} = \mathbf{0}$ outside the body, the surface integral vanishes. The total first moment is thus

$$\mathbf{M} = -\int_V \mathbf{r} \times (\nabla p - \tfrac{1}{2}\nabla \times \mathbf{t}) \, dV$$

Upon comparison with the expression for the external moment of the force (per unit volume) distribution we see that the internal moment is indeed represented by an equivalent macroscopic distribution, the *proper* (D2) pressure gradient being

$$\nabla p^* = \nabla p - \tfrac{1}{2}\nabla \times \mathbf{t} \tag{174}$$

It appears on the basis of our analysis that the proper pressure gradient is a more fundamental entity than is the conventional pressure gradient ∇p.

In order to complete the analysis we require constitutive equations relating both the pressure gradient and the torque density to the local velocity field $\mathbf{q}$. Assuming these relations to be linear, the least restrictive assumption we can make is to suppose that these vectors possess a Taylor series expansion. Thus

$$\nabla p = -\mu(\mathbf{K}_t \cdot \mathbf{q} - \tfrac{1}{2}\mathbf{K}_c' : \nabla\mathbf{q})$$

$$\mathbf{t} = \mu(\mathbf{K}_c'' \cdot \mathbf{q} - \tfrac{1}{2}\mathbf{K}_r : \nabla\mathbf{q})$$

in which it has been assumed that higher-order terms in the expansion may be neglected. The dyadic $\mathbf{K}_t$, pseudodyadic $\mathbf{K}_c''$, triadic $\mathbf{K}_c'$, and pseudotriadic $\mathbf{K}_r$ are assumed to depend only on the geometrical properties of the porous medium. We restrict ourselves further to helicoidally isotropic porous media.

(The usual spherical isotropy is a special case of this.) This requires that we set

$$\mathbf{K}_t = \mathbf{I}K_t, \qquad \mathbf{K}_c{}' = \boldsymbol{\varepsilon}K_c{}'$$

$$\mathbf{K}_c{}'' = \mathbf{I}K_c{}'', \qquad \mathbf{K}_r = \boldsymbol{\varepsilon}K_r$$

where K_t and K_r are true scalars and $K_c{}'$ and $K_c{}''$ are pseudoscalars. Though it is not strictly essential, we shall also *assume* that

$$K_c{}' = K_c{}'' = K_c, \quad \text{say}$$

This reduces from four to three the number of independent phenomenological coefficients required to describe the intrinsic resistance of the porous medium. Inasmuch as $\boldsymbol{\varepsilon}:\nabla\mathbf{q} = -\nabla \times \mathbf{q}$, the constitutive equations thus become

$$\nabla \not{p} = -\mu(K_t\mathbf{q} + K_c\tfrac{1}{2}\nabla \times \mathbf{q}) \tag{175a}$$

$$\mathbf{t} = \mu(K_c\mathbf{q} + K_r\tfrac{1}{2}\nabla \times \mathbf{q}) \tag{175b}$$

In a certain sense these equations are the analogs of the fundamental single particle equations (38)–(39), at least in the isotropic case. [See also Eqs. (109)–(110) for the case where the particle neither translates nor rotates, bearing in mind that $\boldsymbol{\omega}_f = \frac{1}{2}\nabla \times \mathbf{u}$, and that the shear-force and shear-torque triadics vanish for isotropic particles.] It is on the basis of this analogy that we have assumed the symmetry relation $K_c{}' = K_c{}''$.

To the order of the approximation, the local energy dissipation rate (per unit time per unit volume) is

$$E = -\nabla \not{p} \cdot \mathbf{q} + \mathbf{t} \cdot \tfrac{1}{2}\nabla \times \mathbf{q}$$

which from (175) is

$$E = \mu[K_t\mathbf{q}^2 + 2K_c\mathbf{q} \cdot (\tfrac{1}{2}\nabla \times \mathbf{q}) + K_r(\tfrac{1}{2}\nabla \times \mathbf{q})^2]$$

Since the energy dissipation is positive-definite this requires that

$$\det \begin{Vmatrix} K_t & K_c \\ K_c & K_r \end{Vmatrix} > 0$$

and, hence,

$$K_t > 0 \tag{176a}$$

$$K_r > 0 \tag{176b}$$

$$K_tK_r > K_c{}^2 \tag{176c}$$

The scalar coefficient K_t is, of course, the reciprocal of the ordinary permeability k. Two porous media which are statistically equivalent in all respects except screw-sense ought to possess the same K_t, K_r, and $|K_c|$ values, differing only in the algebraic sign of the pseudoscalar K_c. For porous media devoid of screw-like particles (or paths) or consisting of homogeneous. racemic mixtures of screw-like particles it is natural to suppose that $K_c = 0$.

In accordance with Eqs. (174) and (175) the equation of motion is

$$\nabla \not{p}^* = -\mu(K_t \mathbf{q} + \tfrac{1}{2}K_c \nabla \times \mathbf{q} - \tfrac{1}{4}K_r \nabla^2 \mathbf{q}) \tag{177}$$

which is to be considered simultaneously with the continuity relation

$$\nabla \cdot \mathbf{q} = 0 \tag{178}$$

The coefficients K_t, K_c, and K_r are dimensionally of orders a^{-2}, a^{-1}, and a^0, respectively, where a is a characteristic particle dimension (or average spacing between adjacent particles). If we conceive of the porous medium as a discrete collection of touching particles we may then write

$$K_t = a^{-2}\underline{K}_t \tag{179a}$$

$$K_c = a^{-1}\underline{K}_c \tag{179b}$$

$$K_r = \underline{K}_r \tag{179c}$$

where the dimensionless coefficients $\underline{K}_t$, $\underline{K}_c$, and $\underline{K}_r$ depend only on the structure of the porous medium, but not on the size of the particles of which it is comprised. Consider an experiment performed with a porous medium confined within some apparatus of characteristic dimension l (e.g., a circular cylinder of radius l). Equations (177) and (178) may then be expressed in nondimensional form by introducing the substitutions

$$\nabla = l^{-1}\underline{\nabla}$$

$$\mathbf{q} = V_m \underline{\mathbf{q}}$$

$$\not{p}^* = \mu V_m l^{-1} \underline{\not{p}}^*$$

where the underlined quantities are dimensionless, and V_m is a characteristic fluid speed. Upon introducing these and Eq. (179) into Eq. (177), we obtain

$$\underline{\nabla}\underline{\not{p}}^* = -\left[\underline{K}_t\left(\frac{l}{a}\right)^2 \underline{\mathbf{q}} + \tfrac{1}{2}K_c\left(\frac{l}{a}\right)\underline{\nabla} \times \underline{\mathbf{q}} - \tfrac{1}{4}\underline{K}_r \underline{\nabla}^2 \underline{\mathbf{q}}\right]$$

In the limit as $a/l \to 0$ the last two terms on the right become negligibly small compared with the first. In this limit we thus recover the conventional form of Darcy's law. Darcy's law therefore applies strictly only to very finely divided media ($a \to 0$) or to unbounded media ($l \to \infty$). These same dimensional arguments justify retention of only the leading terms in the Taylor expansions (175).[19]

In the absence of screw-like properties ($K_c = 0$), and for the case $a/l \neq 0$, a number of prior investigators (B32, B33, B34, B35, D4c) have proposed an empirical modification of Darcy's law having a form substantially identical to Eq. (177). [See the discussion of these modifications by Hermans (H12a) and Scheidegger (S3, 1st ed., pp. 111–113).] This is done by heuristic arguments beginning with Stokes equations in the presence of a continuous distribution of external volume forces:

$$\nabla^2 \mathbf{v} - \frac{1}{\mu} \nabla p = -\mathbf{f} \tag{180a}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{180b}$$

where **f** denotes the external force per unit volume exerted by the surroundings at a point of the fluid. By invoking Darcy's law in the form

$$\mathbf{f} = -K_t \mathbf{v} \qquad (K_t > 0) \tag{181}$$

they are thus led to the relation

$$\nabla p = -\mu(K_t \mathbf{v} - \nabla^2 \mathbf{v}) \tag{182}$$

which bears an unmistakable resemblance to Eq. (177) (including the algebraic sign of the coefficients) in the absence of screw-like behavior.[20] Depending upon the magnitude of the dimensionless ratio $K_t l^2 = \underline{K}_t (l/a)^2$, Eq. (182) yields Stokes-like flows when l/a is small. Conversely, when l/a is large,

[19] A relationship of the same functional form as Eq. (177) is obtained by assuming that $\nabla \not{p}^*$ possesses a Taylor series expansion of the general form

$$\nabla \not{p}^* = -\mu(\mathbf{K}^{(t)} \cdot \mathbf{q} + \mathbf{K}^{(c)} : \nabla \mathbf{q} + \mathbf{K}^{(r)} \vdots \nabla\nabla \mathbf{q} + \cdots)$$

The unit isotropic Cartesian tensors of ranks two and three are δ_{ij} and ε_{ijk}, respectively, whereas the three unit isotropic tensors of rank four are, respectively. $\delta_{ij}\,\delta_{kl}$, $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, and $\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}$. These, in conjunction with the condition of incompressibility, $\nabla \cdot \mathbf{q} = 0$, thereby lead to following expression for an isotropic porous medium:

$$\nabla \not{p}^* = -\mu(K^{(t)}\mathbf{q} + K^{(c)} \nabla \times \mathbf{q} + K^{(r)} \nabla^2 \mathbf{q} + \cdots)$$

Though such an analysis leads to a relationship formally identical to Eq. (177) it furnishes no insight into the algebraic signs of the coefficients, nor does it suggest that the origin of these additional terms is due in part to the action of local couples within the porous medium.

[20] In the presence of screw-like properties it seems natural to extend Eq. (181) by writing

$$\mathbf{f} = -(K_t \mathbf{v} + K_c \tfrac{1}{2} \nabla \times \mathbf{v})$$

in which case the analogy with Eq. (177) is even more perfect.

Darcy-like flows obtain except in a thin boundary layer near solid walls, where Darcy's equation is unable to satisfy the no-slip boundary condition [cf. Brinkman's treatment (B35) of flow through a porous medium bounded externally by a circular cylinder with solid walls].

Solutions of Eqs. (177)–(178) appear relatively easy to obtain since they require that

$$\nabla^2 \not{p}^* = 0$$

It seems reasonable to demand satisfaction of the no-slip boundary condition $\mathbf{q} = \mathbf{0}$ at solid boundaries. For the practical case where l/a is large ($K_t l/K_c$ and $K_t l^2/K_r$ large), ordinary Darcy flow will prevail except in a thin boundary layer near solid surfaces. Only in this region will the last two terms on the right side of Eq. (177) be of the same order as the leading term.

III. Flow at Small, Nonzero Reynolds Numbers

A. Introduction

Section II dealt in its entirety with Stokes flows. The velocity fields, forces, torques, and the like associated with such flows are independent of the Reynolds number, $R = aV\rho/\mu$. As such, these flows may be regarded as the leading term [i.e., the term of $O(R^0)$] in an asymptotic solution of the Navier–Stokes equations for small Reynolds numbers. In this section we deal with various perturbation schemes for obtaining asymptotic solutions to terms of higher order in R. The schemes are essentially of two kinds—regular and singular perturbations (V1). In the former case, typified by the rotation of a sphere about its axis in an infinite fluid, the Stokes solution provides a uniformly valid zero-order approximation to the solution of the Navier–Stokes equations throughout the *entire* fluid domain. Higher-order terms may then be obtained by conventional perturbation methods, with R as the small perturbation parameter. In the latter case, typified by streaming flow past a sphere in an unbounded fluid, the Stokes solution is no longer a uniformly valid zero-order approximation to the solution of the Navier–Stokes equations throughout the entire flow field. Rather, as Oseen first showed, the approximation becomes invalid at dimensionless distances $\underline{r}$ from the sphere such that $R\underline{r} = O(1)$. In such cases recourse may be had to *singular* perturbation schemes. Van Dyke's book (V1) discusses the application of such "matched asymptotic expansion" techniques to a wide variety of fluid dynamical problems.

With dimensionless variables defined as in Eq. (4), the steady-state Navier–Stokes and continuity equations may be written in the nondimensional forms

$$\underline{\nabla} p - \underline{\nabla}^2 \underline{\mathbf{v}} + R\underline{\mathbf{v}} \cdot \underline{\nabla}\underline{\mathbf{v}} = \mathbf{0} \tag{183}$$

and

$$\underline{\nabla} \cdot \underline{v} = 0 \tag{184}$$

The dimensionless boundary conditions for streaming flow past a sphere are

$$\underline{v} \to \underline{\mathbf{U}} \quad \text{as} \quad \underline{r} \to \infty \tag{185a}$$

$$\underline{v} = \mathbf{0} \quad \text{at} \quad \underline{r} = 1 \tag{185b}$$

The corresponding conditions for the rotating sphere are

$$\underline{v} \to \mathbf{0} \quad \text{as} \quad \underline{r} \to \infty \tag{186a}$$

$$\underline{v} = \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}} \quad \text{at} \quad \underline{r} = 1 \tag{186b}$$

where we have put $V = a\omega$ in Eq. (4).

Assume a perturbation solution of the form

$$\underline{v}(\underline{\mathbf{r}}: R) = \underline{v}_0(\underline{\mathbf{r}}) + R\underline{v}_1(\underline{\mathbf{r}}) + o(R) \tag{187}$$

with a similar expansion for the pressure field. Upon substituting these into the equations of motion and boundary conditions, one finds upon equating terms of order R^0 that the zero-order fields satisfy Stokes equations,

$$\begin{aligned} \underline{\nabla} \underline{p}_0 - \underline{\nabla}^2 \underline{v}_0 &= \mathbf{0} \\ \underline{\nabla} \cdot \underline{v}_0 &= 0 \end{aligned} \tag{188}$$

Furthermore, the zero-order boundary conditions require that

$$\begin{aligned} \underline{v}_0 \to \underline{\mathbf{U}} &\quad \text{as} \quad \underline{r} \to \infty \\ \underline{v}_0 = \mathbf{0} &\quad \text{at} \quad \underline{r} = 1 \end{aligned} \tag{189}$$

in the case of streaming flow past the sphere, and

$$\begin{aligned} \underline{v}_0 \to \mathbf{0} &\quad \text{as} \quad \underline{r} \to \infty \\ \underline{v}_0 = \underline{\boldsymbol{\omega}} \times \underline{\mathbf{r}} &\quad \text{at} \quad \underline{r} = 1 \end{aligned} \tag{190}$$

for rotary motion of the sphere. Hence, the zero-order solutions are merely the appropriate Stokes solutions.

Upon equating terms of order R^1 it is found that the first-order fields satisfy the differential equations

$$\begin{aligned} \underline{\nabla}^2 \underline{v}_1 - \underline{\nabla} \underline{p}_1 &= \underline{v}_0 \cdot \underline{\nabla} \underline{v}_0 \\ \underline{\nabla} \cdot \underline{v}_1 &= 0 \end{aligned} \tag{191}$$

and boundary conditions

$$\underline{v}_1 = \mathbf{0} \quad \text{at both} \quad \underline{r} = 1 \quad \text{and} \quad \underline{r} = \infty \tag{192}$$

Since the zero-order fields are known, these perturbation equations are linear, *inhomogeneous*, Stokes-type equations. Particular integrals are very easy to obtain. To do so we note that Eq. (191) is formally equivalent to Stokes equations in the presence of nonconservative external volume forces. If $\underline{\mathbf{f}} = \mathbf{f}a^2/\mu V$ denotes the dimensionless force per unit volume exerted by the surroundings at a point in the fluid, Stokes equations in nondimensional form become

$$\begin{aligned} \underline{\nabla}^2\underline{\mathbf{v}} - \underline{\nabla}\underline{p} &= -\underline{\mathbf{f}} \\ \underline{\nabla}\cdot\underline{\mathbf{v}} &= 0 \end{aligned} \tag{193}$$

But the fundamental solution of these equations for an isolated point force of strength $\underline{\mathbf{F}} = \mathbf{F}/\mu a V$ at the point $\underline{\mathbf{R}} = \mathbf{0}$ [i.e., $\underline{\mathbf{f}} = \underline{\mathbf{F}}\,\delta(\underline{\mathbf{R}})$, where δ is the Dirac delta function] is (L1, p. 48; L5, p. 605)

$$\begin{aligned} \underline{\mathbf{v}}(\underline{\mathbf{R}}) &= \frac{1}{8\pi|\underline{\mathbf{R}}|}\left(\mathbf{I} + \frac{\underline{\mathbf{R}}\underline{\mathbf{R}}}{|\underline{\mathbf{R}}|^2}\right)\cdot\underline{\mathbf{F}} \\ \underline{p}(\underline{\mathbf{R}}) &= \frac{\underline{\mathbf{R}}}{4\pi|\underline{\mathbf{R}}|^3}\cdot\underline{\mathbf{F}} \end{aligned} \tag{194}$$

Hence, the particular integral of (193) which vanishes when external forces are absent is

$$\underline{\mathbf{v}}(\underline{\mathbf{r}}) = \frac{1}{8\pi}\int\frac{1}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_O|}\left\{\mathbf{I} + \frac{(\underline{\mathbf{r}}-\underline{\mathbf{r}}_O)(\underline{\mathbf{r}}-\underline{\mathbf{r}}_O)}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_O|^2}\right\}\cdot\underline{\mathbf{f}}(\underline{\mathbf{r}})\,d\underline{V}_O \tag{195a}$$

$$\underline{p}(\underline{\mathbf{r}}) = \frac{1}{4\pi}\int\frac{(\underline{\mathbf{r}}-\underline{\mathbf{r}}_O)}{|\underline{\mathbf{r}}-\underline{\mathbf{r}}_O|^3}\cdot\underline{\mathbf{f}}(\underline{\mathbf{r}}_O)\,d\underline{V}_O \tag{195b}$$

where $d\underline{V}_O = d\underline{x}_O\,d\underline{y}_O\,d\underline{z}_O$. Integration is over all of $\underline{\mathbf{r}}_O$ space. By setting $\underline{\mathbf{f}} = -\underline{\mathbf{v}}_O\cdot\underline{\nabla}\underline{\mathbf{v}}_0$ we obtain a particular integral of Eqs. (191).

In order to satisfy the boundary conditions (192) one may add to this particular integral the general solution of the homogeneous Stokes equations [cf. Brenner (B21), where general methods are developed for solving Stokes equations for spherical bodies]. This complementary function together with the particular integral furnish the solution of the perturbation equations (191) and (192).

As Whitehead (W7) showed long ago, no solution of the perturbation equations exists for streaming flow past a sphere or, in fact, for flow past any particle. The source of the difficulty is easily traced. At great distances from the sphere

$$\underline{\mathbf{v}}_0 = \underline{\mathbf{U}} + O(\underline{r}^{-1}) \tag{196}$$

so that $\underline{\mathbf{v}}_0\cdot\underline{\nabla}\underline{\mathbf{v}}_0 = O(\underline{r}^{-2})$. This immediately leads to a particular integral of (191) such that $\underline{\mathbf{v}}_1 = O(\underline{r}^0)$. But the term of $O(\underline{r}^0)$ is not a constant vector—

for if it were it would satisfy the homogeneous, rather than inhomogeneous, Stokes equations. In order to satisfy the boundary condition (192) at infinity we must add to this particular integral a solution of the homogeneous Stokes equations, i.e., the complementary solution. But the only solution of the latter which is of $O(r^0)$ at infinity is a *constant* vector. And this cannot cancel the *variable* vector of $O(r^0)$ arising from the inhomogeneous equations. The nonexistence of solutions of the perturbation equations for streaming flows is known as Whitehead's paradox.

In the case of a rotating sphere the zero-order Stokes solution is (L5, p. 588) $\mathbf{v}_0 = \boldsymbol{\omega} \times \mathbf{r}/r^3$, and so[21]

$$\mathbf{v}_0 = O(r^{-2}) \tag{197}$$

This makes $\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = O(r^{-5})$, which obviously gives rise to a particular integral of the perturbation equations of the form $\mathbf{v}_1 = O(r^{-3})$. There is now no difficulty in finding a complementary function to permit satisfying the boundary conditions (192). Whitehead's paradox does not therefore extend to such rotary motions. These may be handled by the conventional perturbation scheme proposed by Whitehead.

B. Regular Perturbation Methods

Investigations of the perturbation equations for the rotating sphere have been carried out more or less independently by a number of individuals. The first of these appears to be due to Bickley (B5), who obtained a solution correct to the first order in R. His solution indicates an inflow of fluid towards the poles and a corresponding outflow around the equator. Rather interestingly, to this order in R the torque itself is unaffected and continues to be given by the Stokes formula

$$\mathbf{T} = -8\pi\mu a^3 \boldsymbol{\omega}[1 + o(R)]$$

In an important paper, Collins (C14) extended the above analysis to higher powers in R. Here, the series (187) proceeds in integral powers of $R = a^2\omega\rho/\mu$. Collins showed that the torque may be written as an infinite power series involving only *even* powers of R, the first three terms of which are

$$\mathbf{T} = -8\pi\mu a^3 \boldsymbol{\omega}\left[1 + \frac{1}{1200} R^2 - 7.542 \times 10^{-7} R^4 + O(R^6)\right] \tag{198}$$

[21] This rapid rate of attenuation of the velocity field is characteristic of all bodies for which the coupling dyadic vanishes at the center of reaction, providing that the body rotates about an axis through this point. However, it is only for axially symmetric bodies rotating about their symmetry axes that such motions may be stable.

Ovseenko (O5), apparently unaware of Collin's prior work, made a similar analysis and obtained exactly the same R^2 torque correction term. Similar investigations of the rotation of a sphere in an unbounded fluid are given by DiFrancia (D5), Goldshtik (G6), and Khamrui (K9). In a subsequent paper (K10) the latter also investigated the rotation of an ellipsoid of revolution of small ellipticity about its symmetry axis.

Comparable investigations of the rotation of a sphere bounded externally by an outer, concentric sphere are presented by Bratukhin (B9), Haberman (H2), Langlois (L9), and Ovseenko (O6).

As a special case of some results due to Cox (C17), discussed at length in Section III,C, it appears that the torque on *any* body of revolution is unaffected by Reynolds number to the first order in R when it rotates about its symmetry axis. Rather, the torque is affected only in the $O(R^2)$ approximation. Cox also finds that when the body lacks fore-aft symmetry it experiences a *force* of $O(R)$ parallel to the symmetry axis [see Eq. (236)]. No such force appears in the Stokes approximation.

The remarkably small effect of Reynolds number indicated by Eq. (198) is well borne out experimentally. The torque on rotating bodies has been extensively studied in connection with power requirements for mixers, agitators, and similar devices. In the so-called "laminar" regime, logarithmic plots of the dimensionless Power number

$$N_{\text{Po}} = \frac{T}{\mu d^3 \omega}(N_{\text{Re}})^{-1}$$

vs the impeller Reynolds number $N_{\text{Re}} = d^2\omega\rho/\mu$ ($d =$ impeller diameter) are usually linear up to about $N_{\text{Re}} = 10$ (P5), roughly in agreement with the effective linear range indicated by Eq. (198). It should be pointed out, however, that these data pertain to propellers, turbine blades, and paddles, rather than spheres. This writer is not aware of any data on spherical particles with which (198) may be more quantitatively compared.

Conventional perturbation schemes have also found application in the problem of flow through a wavy pipe with corrugated walls (B3). A comparable two-dimensional problem involving shearing flow past a sinusoidal wall was solved long ago by Rayleigh (R2).

C. Singular Perturbation Methods

Whitehead's paradox was resolved by Oseen (O3, O4), who pointed out a fundamental inconsistency in connection with Stokes-type approximations of the streaming flow problem. This lack of internal consistency is evident if we compute the ratio of inertial to viscous terms from Stokes solution (196).

Since $\underline{\nabla} = O(\underline{r}^{-1})$ this ratio is

$$\frac{R|\underline{\mathbf{v}}_0 \cdot \underline{\nabla}\underline{\mathbf{v}}_0|}{|\underline{\nabla}^2\underline{\mathbf{v}}_0|} = \frac{RO(\underline{r}^{-2})}{O(\underline{r}^{-3})} = O(R\underline{r})$$

when $\underline{r}$ is large. Thus, no matter how small R may be, the assumption that the inertial terms are negligible compared with the viscous terms becomes invalid for sufficiently large $\underline{r}$. In particular, the assumption breaks down when $R\underline{r} = O(1)$[22]. Evidently, the *global* criterion that R be small is a necessary but insufficient condition for the success of Stokes-type approximations.

When R is small, Stokes solution is inconsistent only for large $\underline{r}$. But at these large distances the flow is virtually a uniform stream, $\underline{\mathbf{U}}$. Oseen therefore suggested replacing the nonlinear inertial term $\underline{\mathbf{v}} \cdot \underline{\nabla}\underline{\mathbf{v}}$ in the Navier–Stokes equations (183) by the linearized approximation $\underline{\mathbf{U}} \cdot \underline{\nabla}\underline{\mathbf{v}}$. The resulting linear equations,

$$\begin{aligned} \underline{\nabla}p - \underline{\nabla}^2\underline{\mathbf{v}} + R\underline{\mathbf{U}} \cdot \underline{\nabla}\underline{\mathbf{v}} &= \mathbf{0} \\ \underline{\nabla} \cdot \underline{\mathbf{v}} &= 0 \end{aligned} \tag{199}$$

are known as Oseen's equations (O4) (for steady flows). Oseen himself obtained a first-order solution, satisfying the differential equations and boundary condition $\underline{\mathbf{v}} = \underline{\mathbf{U}}$ at $\underline{r} = \infty$ exactly, and the boundary condition $\underline{\mathbf{v}} = \mathbf{0}$ on the sphere $\underline{r} = 1$ correctly to $O(R)$. As is well known, he found the force on the sphere to be given correctly to the zeroth order in R by Stokes law, $\mathbf{F} = 6\pi\mu a\mathbf{U}$.

By removing the inconsistency, Oseen thus placed Stokes result on a firmer mathematical footing. In effect, Oseen's solution provides a uniformly valid zero-order approximation to the solution of the Navier–Stokes equations throughout all space. Unfortunately, a stronger interpretation than this was ascribed to Oseen's solution. Prevailing opinion at the time, and indeed for many years after, held that Oseen's solution was, in fact, an asymptotically valid solution of the Navier–Stokes equations to $O(R)$. This led to the well-known Oseen correction of Stokes law,[23]

$$\mathbf{F} = 6\pi\mu a\mathbf{U}[1 + \tfrac{3}{8}R + o(R)] \tag{200}$$

and occasioned a vast body of literature dealing with the solution of Oseen's equations for a variety of physical situations.

[22] Note from Eq. (197) for a *rotating* sphere that the ratio of inertial to viscous terms is of $O(R\underline{r}^{-2})$. Thus, Stokes assumption is consistent for all $\underline{r}$ when $R \to 0$. In essence, this is the reason why regular perturbation schemes work for the rotating sphere, but are invalid for streaming flow.

[23] Goldstein's (G10) extension of Oseen's solution to $O(R^5)$ was found by Shanks (S9) to be in error in the last term.

Though inconsistencies were discovered in this interpretation,[24] it was not until the work of Lagerstrom, Cole, and Kaplun (K3, K4, L2, L3) and of Proudman and Pearson (P11) that this long-standing issue was resolved by the introduction of singular perturbation techniques. Their work showed that Oseen's analysis was not, in fact, correct to $O(R)$—but only to $O(R^0)$. In its application to low Reynolds number streaming flows the underlying theme of the technique is as follows: One abandons, at least temporarily, attempts to obtain a single, *uniformly valid* expansion of the exact Navier–Stokes equations in orders of R. Instead, one attempts to find *two* expansions of the exact solution, each one of which is only *locally valid* in a different region of the fluid.

The "inner" expansion, valid only in the neighborhood of the particle, is essentially of the type envisioned by Whitehead (W7):

$$\underline{\mathbf{v}}(R;\underline{\mathbf{r}}) = \underline{\mathbf{v}}_0(\underline{\mathbf{r}}) + R\underline{\mathbf{v}}_1(\underline{\mathbf{r}}) + o(R) \tag{201a}$$

$$\underline{p}(R;\underline{\mathbf{r}}) = \underline{p}_0(\underline{\mathbf{r}}) + R\underline{p}_1(\underline{\mathbf{r}}) + o(R) \tag{201b}$$

When these are substituted into the exact Navier–Stokes equations (183)–(184), and into the "inner" boundary condition (185b), and terms of the same order in R equated, one finds that the first few terms in the expansion satisfy the relations

$$\begin{aligned} \underline{\nabla}^2\underline{\mathbf{v}}_0 - \underline{\nabla}\underline{p}_0 &= \mathbf{0} \\ \underline{\nabla}\cdot\underline{\mathbf{v}}_0 &= 0 \\ \underline{\mathbf{v}}_0 &= \mathbf{0} \qquad \text{at} \quad \underline{r} = 1 \end{aligned} \tag{202}$$

and

$$\begin{aligned} \underline{\nabla}^2\underline{\mathbf{v}}_1 - \underline{\nabla}\underline{p}_1 &= \underline{\mathbf{v}}_0\cdot\underline{\nabla}\underline{\mathbf{v}}_0 \\ \underline{\nabla}\cdot\underline{\mathbf{v}}_1 &= 0 \\ \underline{\mathbf{v}}_1 &= \mathbf{0} \qquad \text{at} \quad \underline{r} = 1 \end{aligned} \tag{203}$$

Observe that these equations are linear. Since it is not expected that the inner expansion will be valid at great distances from the sphere, these fields are not required to satisfy the physical boundary conditions at infinity. In the absence of some additional "outer" boundary condition these fields are not unique. The additional boundary condition required for uniqueness is furnished by the "matching" requirement with the outer expansion.

The "outer" expansion, valid only at great distances from the sphere, is

[24] See, for example, the paradox uncovered by Garstang (G1a) and subsequently resolved by Rubinow and Keller (R7).

of a very different character. We first stretch the coordinates by defining the outer coordinates

$$\tilde{\mathbf{r}} = R\mathbf{r} \tag{204}$$

and scale the independent variables by introducing the outer variables

$$\begin{aligned} \tilde{\mathbf{v}}(R;\tilde{\mathbf{r}}) &= \mathbf{v}\left(R;\mathbf{r}=\frac{\tilde{\mathbf{r}}}{R}\right) \\ \tilde{p}(R;\tilde{\mathbf{r}}) &= \frac{1}{R}\,p\left(R;\mathbf{r}=\frac{\tilde{\mathbf{r}}}{R}\right) \end{aligned} \tag{205}$$

In terms of these new quantities the exact Navier–Stokes equations and boundary conditions become

$$\begin{aligned} \tilde{\nabla}^2\tilde{\mathbf{v}} - \tilde{\nabla}\tilde{p} - \tilde{\mathbf{v}}\cdot\tilde{\nabla}\tilde{\mathbf{v}} &= \mathbf{0} \\ \tilde{\nabla}\cdot\tilde{\mathbf{v}} &= 0 \\ \tilde{\mathbf{v}} &= \mathbf{U} \qquad \text{at} \quad \tilde{r} = \infty \\ \tilde{\mathbf{v}} &= \mathbf{0} \qquad \text{at} \quad \tilde{r} = R \end{aligned} \tag{206}$$

These transformations have explicitly removed the Reynolds number from the differential equations of motion, thus making the viscous and inertial terms of comparable order. It was, in fact, to accomplish this (at least in the limiting case $R \to 0$) that the various transformations were originally introduced.

The outer expansion is assumed to be of the form

$$\tilde{\mathbf{v}}(R;\tilde{\mathbf{r}}) = \tilde{\mathbf{v}}_0(\tilde{\mathbf{r}}) + R\tilde{\mathbf{v}}_1(\tilde{\mathbf{r}}) + o(R) \tag{207a}$$

$$\tilde{p}(R;\tilde{\mathbf{r}}) = \tilde{p}_0(\tilde{\mathbf{r}}) + R\tilde{p}_1(\tilde{\mathbf{r}}) + o(R) \tag{207b}$$

Upon substituting these into the equations of motion and outer boundary condition in (206), one finds upon equating terms of equal order in R that the first few terms in the expansion satisfy the relations

$$\begin{aligned} \tilde{\nabla}^2\tilde{\mathbf{v}}_0 - \tilde{\nabla}\tilde{p}_0 - \tilde{\mathbf{v}}_0\cdot\tilde{\nabla}\tilde{\mathbf{v}}_0 &= \mathbf{0} \\ \tilde{\nabla}\cdot\tilde{\mathbf{v}}_0 &= 0 \\ \tilde{\mathbf{v}}_0 &= \mathbf{U} \qquad \text{at} \quad \tilde{r} = \infty \end{aligned} \tag{208}$$

and

$$\begin{aligned} \tilde{\nabla}^2\tilde{\mathbf{v}}_1 - \tilde{\nabla}\tilde{p}_1 - (\tilde{\mathbf{v}}_1\cdot\tilde{\nabla}\tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_0\cdot\tilde{\nabla}\tilde{\mathbf{v}}_1) &= \mathbf{0} \\ \tilde{\nabla}\cdot\tilde{\mathbf{v}}_1 &= 0 \\ \tilde{\mathbf{v}}_1 &= \mathbf{0} \qquad \text{at} \quad \tilde{r} = \infty \end{aligned} \tag{209}$$

Both sets of equations are linear. Since it is not expected that the outer expansion is valid near the body, these fields are not required to satisfy the inner boundary conditions on the body. Accordingly, Eqs. (208) and (209) are not unique without the specification of some "inner" boundary condition. This additional condition is furnished by the "matching" requirement with the inner expansion.

As they presently stand, neither the inner nor outer fields are uniquely defined. The inner fields lack an outer boundary condition for large $\underline{r}$, whereas the outer fields lack an inner boundary condition for small $\underline{\tilde{r}}$. Both sets of conditions are simultaneously furnished by the *matching principle*. According to this principle, two asymptotic solutions of the same differential equations and boundary conditions must be asymptotically equal in their common domain of validity (if such an overlap region exists). The principle is based on the observation that the inner and outer expansions are merely two alternative asymptotic expansions (for small R) of the one *exact* solution of the problem, one being the "Stokes expansion," $R \downarrow 0$ with $\underline{r}$ fixed, and the other being the "Oseen expansion," $R \downarrow 0$ with $R\underline{r}$ fixed. On the basis of earlier observations it is to be expected that the overlap domain in which both solutions are equally valid occurs when $R\underline{r} = O(1)$. This dimensionless disance from the sphere is small in terms of the stretched coordinate $\underline{\tilde{r}} = R\underline{r}$, but it is very large in terms of the unstretched inner coordinate $\underline{r}$. Thus, roughly speaking, the matching condition requires that, when expressed in one common set of coordinates and variables—either inner or outer—

$$\underline{\tilde{\mathbf{v}}}(R; \underline{\tilde{r}} \to 0) \equiv \underline{\mathbf{v}}(R; \underline{r} \to \infty) \tag{210}$$

to any order in R.

Matching to the zeroth order in R requires that

$$\underline{\tilde{\mathbf{v}}}_0(\underline{\tilde{r}} \to 0) \equiv \underline{\mathbf{v}}_0(\underline{r} \to \infty)$$

In conjunction with Eqs. (202) and (207) it is found that the zero-order term in the outer expansion is simply

$$\underline{\tilde{\mathbf{v}}}_0 = \underline{\mathbf{U}} \tag{211}$$

with $\underline{\tilde{p}}_0 = 0$, while the zero-order term, $(\underline{\mathbf{v}}_0, \underline{p}_0)$, in the inner expansion is, as would be expected, Stokes solution for streaming flow past the sphere.

Substitution of (211) into (209) shows that the first-order term in the outer expansion satisfies Oseen's equations (199). The first-order term in the inner expansion, defined by Eq. (203), obviously corresponds to a Whitehead-type term, except that the original difficulty (W7) encountered in attempting to satisfy the actual boundary condition at infinity no longer exists. Final expressions for these fields satisfying the matching conditions to $O(R)$ are given in the original reference (P11). With regard to the force on the sphere, Proudman and Pearson obtained Oseen's result, Eq. (200). That they obtained the same result as did Oseen is, however, fortuitous. For Oseen's original velocity field does not agree to $O(R)$ with the correct asymptotic solution of the Navier–Stokes equations. Rather it is correct only to $O(R^0)$.

Continuing their approximation, Proudman and Pearson (P11) find

$$\mathbf{F} = 6\pi\mu a\mathbf{U}[1 + \tfrac{3}{8}R + \tfrac{9}{40}R^2 \ln R + O(R^2)] \tag{212}$$

The appearance of transcendental terms in R is in marked contrast to Goldstein's (G10) formula for the force on a sphere to $O(R^5)$ (see footnote 23) when the Oseen, rather than Navier–Stokes, equations are taken as the basic equations of motion. Equation (212) represents an asymptotic expansion having the general form

$$\mathbf{F} = 6\pi\mu a\mathbf{U}[f_0(R) + f_1(R) + f_2(R) + \cdots]$$

where $f_0(R) = 1$ and, for $n = 0, 1, 2, \ldots,$

$$f_{n+1}(R)/f_n(R) \to 0 \qquad \text{as} \quad R \to 0$$

An expression of the same general structure as (212) was obtained by Breach (B10) for axially symmetric streaming flow past prolate and oblate spheroids. In a significant extension of Breach's analysis, Shi (S9a) treats the problem of streaming flow past a prolate spheroid of large aspect ratio for the case where the axis of revolution of the ellipsoid is perpendicular, rather than parallel, to the uniform stream at infinity. The observation is made that such an ellipsoidal "needle" possesses *two* characteristic lengths (the major and minor semi-axes), each of which gives rise to its own distinct Reynolds number. In contrast with Breach's treatment, where it is implicitly assumed that both of these Reynolds numbers are small compared with unity, Shi analyzes the case where the minor-axis Reynolds number tends to zero for a *fixed* value of the major-axis Reynolds number. The latter need not be small. Though the ellipsoidal needle is a strictly three-dimensional particle, the drag per unit length is found to be the same as for two-dimensional streaming flow past a circular cylinder (K3, K4, P11), at least to terms of lowest order in the minor-axis Reynolds number. As such, the asymptotic expansion formula for the drag proceeds in inverse powers of the logarithm of the minor-axis Reynolds number, in marked contrast with formulas of the type (212). The coefficients of these logarithmic terms explicitly involve the major-axis Reynolds number. Shi's analysis clarifies the relationship between Oberbeck's classical solution for Stokes flow past an ellipsoid [see Eq. (58)] of large aspect ratio and two-dimensional "Oseen" flow past a circular cylinder.

Using these asymptotic matching techniques, Rubinow and Keller (R7) calculated, to $O(R)$, the force and torque on a translating sphere simultaneously spinning about any axis through its center in a fluid at rest at infinity.

They found for the force

$$\mathbf{F} = -6\pi\mu a\mathbf{U}(1 + \tfrac{3}{8}R) + R\,\frac{\pi\mu a^2}{U}\,\boldsymbol{\omega} \times \mathbf{U} + \mu a U o(R) \qquad (213)$$

and, for the torque about the sphere center,

$$\mathbf{T} = -8\pi\mu a^3\boldsymbol{\omega}[1 + O(R)] \qquad (214)$$

The hydrodynamic force thus consists of a drag force,

$$\mathbf{F}_D = -6\pi\mu a\mathbf{U}[1 + \tfrac{3}{8}R + o(R)] \qquad (215)$$

antiparallel to the direction of motion of the sphere, and a lift force,

$$\mathbf{F}_L = \pi a^3\rho\boldsymbol{\omega} \times \mathbf{U}[1 + O(R)] \qquad (216)$$

at right angles to this direction. Rather remarkably, this sidewise force is independent of viscosity for small values of R. This force is comparable to the well-known Magnus force (G11), arising at very high Reynolds numbers, usually invoked to explain such phenomena as the curving of a pitched baseball.

The solution of the Rubinow–Keller problem had previously been attempted by Garstang (G1a) on the basis of the Oseen equations. His result for the lift force is larger than (216) by a factor of 4/3. But as Garstang himself pointed out, his result was not unequivocal. Rather, different results were obtained according as the integration of the momentum flux was carried out at the surface of the sphere or at infinity. Garstang's "paradox" is clearly due to the fact that the term $\mathbf{U}\cdot\nabla\mathbf{v}$ does not represent a uniformly valid approximation of the inertial term $\mathbf{v}\cdot\nabla\mathbf{v}$ throughout all portions of the fluid, at least not to the first order in R.

To terms of $O(R)$, Brenner (B15) and Chester (C7) independently[25] extended the Proudman and Pearson analysis (P11) to streaming flow past the general class of bodies possessing sufficient symmetry to cause the hydrodynamic vector force on the body to be reversed when the direction of the stream velocity at infinity is reversed without change of magnitude, e.g., an ellipsoid at an arbitrary angle of incidence. A simple algebraic expression was found relating this force directly to the Stokes translation dyadic for the body. (This general formula, as well as those discussed in subsequent paragraphs, is applicable only to the case where the Reynolds number based upon the *maximum* linear dimension of the particle is small compared with unity.)

These results were further generalized by Brenner and Cox (B27) to rigid bodies of arbitrary shape, devoid of any symmetry whatsoever, and extended

[25] These results to $O(R)$ were also obtained independently by O'Brien (O1) in the special case of axially symmetric flow past a body of revolution possessing fore-aft symmetry.

by them to terms of $O(R^2 \ln R)$. Finally, to this same order in R, Cox (C17) generalized the latter analysis to the case where the body in question may also rotate as it translates. In particular, general expressions were obtained for the force and torque, to $O(R^2 \ln R)$, experienced by a rigid translating-rotating body of arbitrary shape in steady or quasi-steady motion through a fluid at rest at infinity. The results are expressed explicitly in terms of the intrinsic translational and rotational *Stokes* dyadic velocity fields (see Section II,C,1). Thus, detailed knowledge of the translational and rotational flow fields to $O(R^0)$ permits an immediate computation of the force and torque on the body in question to $O(R^2 \ln R)$. The calculation itself involves only a straightforward, though perhaps tedious, quadrature. Moreover, if the body possesses certain geometric symmetries, one need not even know the detailed Stokes velocity fields for the body. Rather, the forces and torques to $O(R^2 \ln R)$ may be computed, without quadrature, solely from knowledge of the Stokes translation, rotation, and coupling dyadics for the body in question. And these are purely phenomenological coefficients which, at least in principle, may be obtained from purely macroscopic experiments on the body in the Stokes regime. One by-product of this investigation (B27, C17) is that Oseen's equation, though furnishing the correct vector force to $O(R)$ on bodies possessing a high degree of symmetry, e.g., spheres and ellipsoids, generally leads to an incorrect result to $O(R)$ for less symmetrical bodies.

As the Brenner–Cox (B27, C17) relations are not readily susceptible to further generalization it is appropriate to summarize them in some detail here. A nondimensional form is most convenient. As usual, let $\mathbf{U}$ be the velocity of any point O affixed to the body and $\boldsymbol{\omega}$ the angular velocity of the body. Further, let a be any characteristic particle dimension and $\mathscr{V}$ be any characteristic speed, e.g., $|\mathbf{U}|$ or $a|\boldsymbol{\omega}|$. The ultimate results are independent of the choice of these characteristic parameters. $R = a\mathscr{V}\rho/\mu$ denotes the particle Reynolds number. A dimensionless hydrodynamic force and torque about O are defined as follows:

$$\begin{aligned} \underline{\mathbf{F}} &= \mathbf{F}/6\pi\mu a\mathscr{V} \\ \underline{\mathbf{T}} &= \mathbf{T}/6\pi\mu a^2\mathscr{V} \end{aligned} \tag{217}$$

Introduction of the extraneous factor of 6π has the advantage of rendering many of the subsequent coefficients unity for translating spherical particles. The vectors

$$\underline{\mathbf{U}} = \mathbf{U}/\mathscr{V}, \qquad \underline{\boldsymbol{\omega}} = \boldsymbol{\omega} a/\mathscr{V} \tag{218}$$

denote the dimensionless particle velocity and spin. Note carefully that these are not unit vectors in general, though either one could be made so by an appropriate choice of $\mathscr{V}$. Furthermore, let

$$\overset{(t)}{\underline{\mathbf{K}}} = \overset{(t)}{\mathbf{K}}/6\pi a$$
$$\overset{(c)}{\underline{\mathbf{K}}} = \overset{(c)}{\mathbf{K}}/6\pi a^2 \tag{219}$$
$$\overset{(r)}{\underline{\mathbf{K}}} = \overset{(r)}{\mathbf{K}}/6\pi a^3$$

represent the dimensionless Stokes translation, coupling, and rotation dyadics at O. Finally, set

$$\overset{(t)}{\underline{\mathbf{V}}} = \overset{(t)}{\mathbf{V}}, \qquad \overset{(r)}{\underline{\mathbf{V}}} = \overset{(r)}{\mathbf{V}}/a \tag{220}$$

where $\overset{(t)}{\mathbf{V}}$ and $\overset{(r)}{\mathbf{V}}$ are the (dimensional) intrinsic Stokes translational and rotational dyadic " velocity " fields defined in Section II,C,1. The corresponding dimensionless rate-of-strain triadic fields are denoted by

$$\overset{(\,)}{\underline{\mathbf{E}}} = \tfrac{1}{2}\left[\underline{\nabla}\overset{(\,)}{\underline{\mathbf{V}}} + {}^{\dagger}\!\left(\underline{\nabla}\overset{(\,)}{\underline{\mathbf{V}}}\right)\right] \tag{221}$$

in which $\underline{\nabla} = a\nabla$.

The principal results are as follows:

(i) The force and torque on a solid body of arbitrary shape moving with steady or quasi-steady[26] motion through an incompressible fluid at rest at infinity are given to $O(R^2 \ln R)$ by the expressions

$$\begin{Bmatrix} \underline{\mathbf{F}} \\ \underline{\mathbf{T}} \end{Bmatrix} = \begin{Bmatrix} {}^1\underline{\mathbf{F}} \\ {}^1\underline{\mathbf{T}} \end{Bmatrix} + \begin{Bmatrix} {}^2\underline{\mathbf{F}} \\ {}^2\underline{\mathbf{T}} \end{Bmatrix} + O(R^2) \tag{222a, 222b}$$

Here,

$$\begin{Bmatrix} {}^1\underline{\mathbf{F}} \\ {}^1\underline{\mathbf{T}} \end{Bmatrix} = \begin{Bmatrix} \underline{\mathbf{F}}_0 \\ \underline{\mathbf{T}}_0 \end{Bmatrix} - \begin{Bmatrix} \overset{(t)}{\underline{\mathbf{K}}} \\ \overset{(c)}{\underline{\mathbf{K}}} \end{Bmatrix} \cdot \left[\frac{3}{16} R \frac{1}{|\underline{\mathbf{U}}|} (3\underline{\mathbf{F}}_0 \underline{\mathbf{U}} \cdot \underline{\mathbf{U}} - \underline{\mathbf{F}}_0 \cdot \underline{\mathbf{U}}\underline{\mathbf{U}})\right.$$
$$\left. + \frac{3}{320} (R^2 \ln R)(31\, \underline{\mathbf{F}}_0 \underline{\mathbf{F}}_0 \cdot \underline{\mathbf{U}} - 7 \underline{\mathbf{U}}\underline{\mathbf{F}}_0 \cdot \underline{\mathbf{F}}_0)\right] \tag{223a, 223b}$$

in which

$$\begin{Bmatrix} \underline{\mathbf{F}}_0 \\ \underline{\mathbf{T}}_0 \end{Bmatrix} = - \begin{Bmatrix} \overset{(t)}{\underline{\mathbf{K}}} \\ \overset{(c)}{\underline{\mathbf{K}}} \end{Bmatrix} \cdot \underline{\mathbf{U}} - \begin{Bmatrix} \overset{(c)}{\underline{\mathbf{K}}}{}^{\dagger} \\ \overset{(r)}{\underline{\mathbf{K}}} \end{Bmatrix} \cdot \underline{\boldsymbol{\omega}} \tag{224, 225}$$

are, respectively, the Stokes force and torque on the body.

[26] By quasi-steady motion here, we mean an unsteady motion for which the time-dependent terms in the equations of motion and boundary conditions are of higher order in R than $O(R^2 \ln R)$.

The remaining terms are given by the expressions

$$\begin{Bmatrix} {}^2\underline{\mathbf{F}} \\ {}^2\underline{\mathbf{T}} \end{Bmatrix} = \frac{R}{6\pi}\left[\begin{Bmatrix} \overset{(t)}{\underline{\mathcal{K}}}{}^{[tt]} \\ \overset{(r)}{\underline{\mathcal{K}}}{}^{[tt]} \end{Bmatrix} : \underline{\mathbf{U}}\underline{\mathbf{U}} + 2\begin{Bmatrix} \overset{(t)}{\underline{\mathcal{K}}}{}^{[rt]} \\ \overset{(r)}{\underline{\mathcal{K}}}{}^{[rt]} \end{Bmatrix} : \underline{\mathbf{U}}\boldsymbol{\omega} + \begin{Bmatrix} \overset{(t)}{\underline{\mathcal{K}}}{}^{[rr]} \\ \overset{(r)}{\underline{\mathcal{K}}}{}^{[rr]} \end{Bmatrix} : \underline{\boldsymbol{\omega}\boldsymbol{\omega}}\right] \qquad \begin{matrix}(226a)\\ \\ (226b)\end{matrix}$$

in which the $\mathcal{K}$'s are intrinsic, constant, dimensionless, inertial resistance triadics, dependent solely upon the external shape of the body. Written in Cartesian tensor form[27] they may be expressed in terms of the detailed translational and rotational Stokes fields for the body in question by means of the relations

$$\overset{(\ )}{\underline{\mathcal{K}}}{}^{[rr]}_{jkl} = \int_{\underline{\Gamma}} \overset{(\ )}{\underline{E}}_{mnj}\overset{(r)}{\underline{V}}_{mk}\overset{(r)}{\underline{V}}_{nl}\, d\underline{\Gamma} \tag{227a}$$

$$\overset{(\ )}{\underline{\mathcal{K}}}{}^{[rt]}_{jkl} = \int_{\underline{\Gamma}} \overset{(\ )}{\underline{E}}_{mnj}\overset{(r)}{\underline{V}}_{mk}\left(\overset{(t)}{\underline{V}}_{nl} - \delta_{nl}\right) d\underline{\Gamma} \tag{227b}$$

$$\begin{aligned}\overset{(\ )}{\underline{\mathcal{K}}}{}^{[tt]}_{jkl} = {} & -\tfrac{1}{2}\left[\int_{\underline{S}_L} {}_{-2}\overset{(\ )}{\underline{V}}_{kj}\, d\underline{S}_l + \int_{\underline{S}_L} {}_{-2}\overset{(\ )}{\underline{V}}_{lj}\, d\underline{S}_k\right] \\ & + \int_{\underline{\Gamma}} \overset{(\ )}{\underline{E}}_{mnj}\left(\overset{(t)}{\underline{V}}_{mk} - \delta_{mk}\right)\left(\overset{(t)}{\underline{V}}_{nl} - \delta_{nl}\right) d\underline{\Gamma}\end{aligned} \tag{227c}$$

which holds for $(\ ) \equiv (t)$ or (r). In these expressions $\underline{\Gamma}$ is the volume of space external to the particle, $\underline{S}_L$ is a spherical surface of indefinitely large radius containing the particle at its center, and ${}_{-2}\underline{\mathbf{V}}$ is that term in $\underline{\mathbf{V}}$ which is homogeneous in $\underline{r}^{-2}$ in the asymptotic expansion of $\underline{\mathbf{V}}$ for large $\underline{r}$. The distance $\underline{r}$, surface $\underline{S}_L$, and volume $\underline{\Gamma}$ have been made dimensionless with the particle dimension a.

We note from Eq. (221) that

$$\overset{(\ )}{\underline{E}}_{mnj} = \tfrac{1}{2}\left(\overset{(\ )}{\underline{V}}_{nj,m} + \overset{(\ )}{\underline{V}}_{mj,n}\right)$$

(ii) The tensors $\overset{(t)}{\underline{\mathcal{K}}}{}^{[tt]}_{ijk}$, $\overset{(t)}{\underline{\mathcal{K}}}{}^{[rr]}_{ijk}$, and $\overset{(r)}{\underline{\mathcal{K}}}{}^{[rt]}_{ijk}$ are *true* (polar) third-order tensors.

[27] That is, if $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are any triad of orthonormal vectors, constant throughout space, then

$$\mathbf{d} = \mathbf{i}_j\mathbf{i}_k d_{jk} \qquad \text{(summation convention)}$$

$$\mathbf{t} = \mathbf{i}_j\mathbf{i}_k\mathbf{i}_l t_{jkl} \qquad \text{(summation convention)}$$

where $\mathbf{d}$ is any dyadic and $\mathbf{t}$ any triadic.

The two "translational" (t)-tensors satisfy the trivial symmetry conditions

$$\overset{(t)}{\underline{\mathscr{K}}}{}^{[tt]}_{ijk} = \overset{(t)}{\underline{\mathscr{K}}}{}^{[tt]}_{ikj} \tag{228a}$$

$$\overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]}_{ijk} = \overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]}_{ikj} \tag{228b}$$

The tensors $\overset{(r)}{\underline{\mathscr{K}}}{}^{[rr]}_{ijk}$, $\overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]}_{ijk}$, and $\overset{(t)}{\underline{\mathscr{K}}}{}^{[rt]}_{ijk}$ are *pseudo* (axial) third-order tensors. The two "rotational" (r)-tensors satisfy the trivial symmetry conditions

$$\overset{(r)}{\underline{\mathscr{K}}}{}^{[rr]}_{ijk} = \overset{(r)}{\underline{\mathscr{K}}}{}^{[rr]}_{ikj} \tag{229a}$$

$$\overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]}_{ijk} = \overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]}_{ikj} \tag{229b}$$

The dependence of these six inertial resistance triadics on choice of origin may be obtained by application of the general methods described in the paragraph immediately following Eq. (50). [See also the similar calculations in Section 3 of Brenner (B23).]

(iii) The ${}^1\underline{\mathbf{F}}$ and ${}^1\underline{\mathbf{T}}$ terms in Eq. (222) can be computed solely from knowledge of the three phenomenological Stokes resistance dyadics. The ${}^2\underline{\mathbf{F}}$ and ${}^2\underline{\mathbf{T}}$ terms require detailed knowledge of the Stokes velocity fields themselves.[28] In dimensional form the ${}^2\mathbf{F}$ and ${}^2\mathbf{T}$ terms are independent of viscosity, and are directly proportional to the fluid density.

The results given here apply to the case where the particle translates (and rotates) in a fluid at rest at infinity. To convert them to the case of streaming flow past the body with stream velocity $\underline{\mathbf{U}}$, as in Brenner and Cox (B27), replace $\underline{\mathbf{U}}$ in the preceding equations by $-\underline{\mathbf{U}}$, and $\overset{(t)}{\underline{\mathbf{V}}}$ by $\mathbf{I} - \overset{(t)}{\underline{\mathbf{V}}}$. Note that the latter substitution requires that $\overset{(t)}{\underline{\mathbf{E}}}$ be replaced by $-\overset{(t)}{\underline{\mathbf{E}}}$.

(iv) If the vectors $\underline{\mathbf{U}}$ and $\underline{\boldsymbol{\omega}}$ are *both* reversed, then ${}^1\underline{\mathbf{F}}$ and ${}^1\underline{\mathbf{T}}$ are reversed, whereas ${}^2\underline{\mathbf{F}}$ and ${}^2\underline{\mathbf{T}}$ remain unaltered.

(v) If $\{\underline{\mathbf{U}}, \underline{\boldsymbol{\omega}}\} \neq \mathbf{0}$, the {force, torque}, $\{{}^2\underline{\mathbf{F}}, {}^2\underline{\mathbf{T}}\}$, may be written as the sum of a *lift* term

$$\begin{Bmatrix} {}^2\underline{\mathbf{F}}_L \\ {}^2\underline{\mathbf{T}}_L \end{Bmatrix} = \frac{R}{6\pi} \begin{Bmatrix} (\mathbf{I} - \underline{\mathbf{U}}\underline{\mathbf{U}}/|\underline{\mathbf{U}}|^2) \\ (\mathbf{I} - \underline{\boldsymbol{\omega}}\underline{\boldsymbol{\omega}}/|\underline{\boldsymbol{\omega}}|^2) \end{Bmatrix} \cdot \left[\begin{Bmatrix} \overset{(t)}{\underline{\mathscr{K}}}{}^{[tt]} \\ \overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]} \end{Bmatrix} : \underline{\mathbf{U}}\underline{\mathbf{U}} \right.$$

$$\left. + 2 \begin{Bmatrix} \overset{(t)}{\underline{\mathscr{K}}}{}^{[rt]} \\ \overset{(r)}{\underline{\mathscr{K}}}{}^{[rt]} \end{Bmatrix} : \underline{\mathbf{U}}\underline{\boldsymbol{\omega}} + \begin{Bmatrix} \overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]} \\ \overset{(r)}{\underline{\mathscr{K}}}{}^{[rr]} \end{Bmatrix} : \underline{\boldsymbol{\omega}}\underline{\boldsymbol{\omega}} \right] \qquad \begin{matrix} (230a) \\ (230b) \end{matrix}$$

[28] One could, of course, alternatively regard the $\underline{\mathscr{K}}$ coefficients as intrinsic phenomenological parameters to be determined from macroscopic experiments on the particle at small, but nonzero Reynolds numbers.

and a *drag* term

$$\begin{Bmatrix} {}^2\underline{\mathbf{F}}_D \\ {}^2\underline{\mathbf{T}}_D \end{Bmatrix} = -\frac{R}{6\pi}\begin{Bmatrix} \underline{\mathbf{U}\boldsymbol{\omega}}/|\underline{\mathbf{U}}|^2 \\ \underline{\boldsymbol{\omega}\mathbf{U}}/|\underline{\boldsymbol{\omega}}|^2 \end{Bmatrix} \cdot \left[\begin{Bmatrix} \overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]} \\ \overset{(t)}{\underline{\mathscr{K}}}{}^{[tt]} \end{Bmatrix} : \underline{\mathbf{UU}} \right.$$

$$\left. + 2\begin{Bmatrix} \overset{(r)}{\underline{\mathscr{K}}}{}^{[rt]} \\ \overset{(t)}{\underline{\mathscr{K}}}{}^{[rt]} \end{Bmatrix} : \underline{\mathbf{U}\boldsymbol{\omega}} + \begin{Bmatrix} \overset{(r)}{\underline{\mathscr{K}}}{}^{[rr]} \\ \overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]} \end{Bmatrix} : \underline{\boldsymbol{\omega\omega}} \right] \qquad \begin{matrix} (231a) \\ (231b) \end{matrix}$$

the lift term being perpendicular to $\{\underline{\mathbf{U}}, \underline{\boldsymbol{\omega}}\}$, and the drag term parallel to $\{\underline{\mathbf{U}}, \underline{\boldsymbol{\omega}}\}$.

(vi) For nonrotating bodies ${}^2\underline{\mathbf{F}}_D$ vanishes, so that ${}^2\underline{\mathbf{F}}$ is purely a *lift* force. It follows that the *drag* on a body of arbitrary shape is thus given to $O(R^2 \ln R)$ by that component of ${}^1\underline{\mathbf{F}}$ lying parallel to $\underline{\mathbf{U}}$. This drag is reversed by a reversal of $\underline{\mathbf{U}}$, and is expressible entirely in terms of the Stokes force $\underline{\mathbf{F}}_0$ on the body. The dimensional drag formula can be expressed in elementary form by setting $\mathscr{V} = |\mathbf{U}|$ and $\mathbf{U} = \boldsymbol{\alpha} U$, where $\boldsymbol{\alpha}$ is a unit vector parallel to the direction of motion. This yields for the drag on any translating body[29]

$$|F_\alpha| = |(F_0)_\alpha| + \tfrac{3}{16}R\,\frac{[3(F_0)^2 - \{(F_0)_\alpha\}^2]}{6\pi\mu a U} + \tfrac{9}{40}(R^2 \ln R)\,\frac{|(F_0)_\alpha|(F_0)^2}{(6\pi\mu a U)^2} + \mu a U O(R^2) \qquad (232)$$

where the subscript α refers to the α component of the corresponding vector force, i.e., $F_\alpha = \mathbf{F}\cdot\boldsymbol{\alpha}$.

If, in addition, the orientation of the body is parallel to a Stokes principal axis of translation, the particle experiences no lift force in Stokes flow, whence $(F_0)_\alpha = F_0$. Hence, the drag on the particle is

$$\frac{F_\alpha}{F_0} = 1 + \frac{3}{8}\left(\frac{|F_0|}{6\pi\mu a U}\right)R + \frac{9}{40}\left(\frac{F_0}{6\pi\mu a U}\right)^2 R^2 \ln R + O(R^2) \qquad (233)$$

As indicated by the notation, the *vector* force on the particle at higher Reynolds numbers need not be parallel to the direction of motion, even though

[29] Actually, one obtains

$$F_\alpha = (F_0)_\alpha - \tfrac{3}{16}R\,\frac{[3(F_0)^2 - \{(F_0)_\alpha\}^2]}{6\pi\mu a U} + \tfrac{9}{40}(R^2 \ln R)\,\frac{(F_0)_\alpha(F_0)^2}{(6\pi\mu a U)^2} + \mu a U O(R^2)$$

But since the F's (i.e., the components of the corresponding vector forces $\mathbf{F}$) are *negative* quantities, the alternative form in Eq. (232) is less misleading.

it is in Stokes flow for the assumed circumstances. By setting $F_0 = -6\pi\mu a U$, Eq. (233) reproduces Proudman and Pearson's (P11) result for the sphere, Eq. (212).

Upon putting $R = aU\rho/\mu$ in (233) it adopts the form[30]

$$\frac{F_\alpha}{F_0} = 1 + \frac{|F_0|\rho}{16\pi\mu^2} + \frac{8}{5}\left(\frac{F_0\rho}{16\pi\mu^2}\right)^2 \ln\left(\frac{aU\rho}{\mu}\right) + O(R^2) \tag{234}$$

In this form it is somewhat more obvious that the term of $O(R)$ here, and in the more general result, is indeed independent of the choice of the characteristic parameters, a and $\mathscr{V}$. On the other hand, the term of $O(R^2 \ln R)$ does depend weakly on the explicit definition of these parameters. This shows that the terms of $O(R^2 \ln R)$ and $O(R^2)$ are, in a sense, inseparable.

(vii) The inertial resistance triadics $\underline{\mathscr{K}}$ may be regarded as purely phenomenological coefficients to be determined from macroscopic experiments. For bodies possessing various types of geometric symmetry, the number and nature of independent tensor components of each of these resistance tensors is appropriately decreased. Systematic calculations of such symmetry reductions, based on the properties set forth in Paragraph (ii), are tabulated for some common types of symmetry in Refs. (B27), (C17), and (B23), though the latter investigation was originally made in a very different context.

Some of the more important examples of these symmetry restrictions and applications thereof are set forth in the following paragraphs.

(viii) If, for a nonrotating particle, body symmetry demands that a certain component of the vector force be reversed to all orders in R for a reversal of $\underline{U}$, then this component is given to $O(R^2 \ln R)$ by the corresponding component of ${}^1\underline{F}$ in Eq. (223a). An ellipsoidal particle at an arbitrary angle of incidence is an example of such a body. In this case Eq. (223a) gives the *complete* vector force on the ellipsoid.

(ix) A nonrotating particle for which the coupling dyadic vanishes at the center of reaction experiences a couple about this point given by [31]

$$\underline{T} = R\frac{1}{6\pi}\overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]} : \underline{U}\underline{U} + O(R^2) \tag{235}$$

[30] The coefficient 8/5 may be written as $1/(1 - 3/8)$. Because of the frequent occurrence of the 3/8 coefficient at various places in the theory, e.g., in Eq. (233), this is thought to be more than coincidental.

[31] In dimensional form,

$$\mathrm{T} = \rho\overset{(r)}{\mathscr{K}}{}^{[tt]} : \mathrm{UU} + \mu a^2 U O(R^2)$$

where $\overset{(r)}{\mathscr{K}}{}^{[tt]} = a^3\overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]}$.

A translating ellipsoid, for example, thus experiences a couple of $O(R)$ unless it moves normal to one of its three symmetry planes, in which case no couple acts to any order in R. By evaluating a key component of $\overset{(r)}{\underline{\mathscr{K}}}{}^{[tt]}$ for the slightly deformed spheroid of revolution

$$\underline{r} = 1 + \varepsilon P_2(\cos\theta) + O(\varepsilon^2)$$

Cox (C17) solved the pertinent equations describing the terminal motion of the spheroid. Cox's calculation reveals that a homogeneous ellipsoid of revolution, albeit of small eccentricity, if allowed to fall freely through the fluid ultimately takes up a stable orientation such that its axis is horizontal if prolate ($\varepsilon < 0$) and vertical if oblate ($\varepsilon > 0$). Thus, the terminal orientation is such as to make the resistance to motion a maximum and, consequently, the settling velocity a minimum. This conclusion is well supported by experimental data in the "transition" range of Reynolds numbers (H12, K11, M4, P6), as is the converse theoretical conclusion (B18, B22) that, in the absence of inertial effects, a homogeneous body possessing three orthogonal planes of reflection symmetry is neutrally stable in *any* orientation.

(x) If a nontranslating body of revolution rotates about its symmetry axis (the Ox_1 axis, say) with angular speed $\underline{\omega}_1$ the components of the force and couple are[32]

$$\underline{F}_1 = R\,\frac{1}{6\pi}\,\overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]}_{111}\,\underline{\omega}_1{}^2 + O(R^2) \tag{236}$$

$$\underline{T}_1 = -\overset{(r)}{\underline{K}}_{11}\underline{\omega}_1 + O(R^2) \tag{237}$$

The other components are zero to all orders in R. That the lowest-order correction to the Stokes couple is of $O(R^2)$ is in agreement with Collins (C14) formula for the rotating sphere, cited in Eq. (198). Equation (236) shows that the rotating body experiences a force directed along the symmetry axis, and that this force is unaltered by a reversal of the angular velocity vector. If the body also possesses fore-aft symmetry, then $\overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]}_{111} = 0$, whence no force results. In the absence of such symmetry the force clearly arises from an unequal influx of fluid towards the poles of the body.

In accordance with the discussion of Section III,B it seems likely that Eqs. (236) and (237) could, perhaps, be more simply obtained by regular, rather than singular, perturbation procedures. That the former technique

[32] In dimensional form the force is

$$F_1 = \rho\,\overset{(t)}{\mathscr{K}}{}^{[rr]}_{111}\,\omega_1{}^2 + \mu a^2\omega_1 O(R^2)$$

where $\overset{(t)}{\mathscr{K}}{}^{[rr]}_{111} = a^4\,\overset{(t)}{\underline{\mathscr{K}}}{}^{[rr]}_{111}$ and $R = a^2\omega_1\rho/\mu$.

should work despite the force experienced by the body can be seen by the following reasoning: If the fluid is at rest at infinity, but the body experiences a force, the Stokes velocity field at great distances is of $O(\underline{r}^{-1})$ (C17). Thus, as $\underline{r} \to \infty$,

$$\frac{R|\underline{\mathbf{v}}_0 \cdot \underline{\nabla}\underline{\mathbf{v}}_0|}{|\underline{\nabla}^2 \underline{\mathbf{v}}_0|} = \frac{R O(\underline{r}^{-3})}{O(\underline{r}^{-3})} = O(R)$$

which is uniformly small.

(xi) Consider an *isotropic* particle moving through the fluid. Though the forms of the six resistance triadics in Eq. (226) can be established by formal symmetry arguments (B23, B27, C17) it is instructive to proceed by less rigorous invariance arguments. Since the particle is isotropic so must be the triadics. But the only isotropic tensor of third order is the alternating triadic $\boldsymbol{\varepsilon}$ (A5). The six resistance triadics must therefore be scalar or pseudoscalar multiples of $\boldsymbol{\varepsilon}$. Since $\boldsymbol{\varepsilon} : \underline{\mathbf{U}}\underline{\mathbf{U}} = -\underline{\mathbf{U}} \times \underline{\mathbf{U}} = \mathbf{0}$ (with a similar identity for $\boldsymbol{\omega}$), one may conclude that the four "direct" triadics in (226) cannot contribute to the final results and, hence, may be taken to be identically zero. Only the "cross" triadics remain. Consequently, at the center of reaction,

$$^{2}\underline{\mathbf{F}} = R \frac{1}{3\pi} \underline{\overset{(t)}{\mathscr{K}}}{}^{[rt]} \underline{\mathbf{U}} \times \underline{\boldsymbol{\omega}} \tag{238}$$

$$^{2}\underline{\mathbf{T}} = R \frac{1}{3\pi} \underline{\overset{(r)}{\mathscr{K}}}{}^{[rt]} \underline{\mathbf{U}} \times \underline{\boldsymbol{\omega}} \tag{239}$$

where $\underline{\overset{(t)}{\mathscr{K}}}{}^{[rt]}$ is a true scalar and $\underline{\overset{(r)}{\mathscr{K}}}{}^{[rt]}$ is a pseudoscalar.[33] Thus, if the particle is *spherically isotropic* we require that $\underline{\overset{(r)}{\mathscr{K}}}{}^{[rt]} = 0$, and the body experiences only a Magnus-type lift force of the form found by Rubinow and Keller (R7) for the sphere.[34] On the other hand, if the body is *helicoidally isotropic* the pseudoscalar coefficient need not vanish. In this case the body also experiences a "lifting" couple at right angles to $\underline{\mathbf{U}}$ and $\underline{\boldsymbol{\omega}}$, the sense of the couple depending upon whether the isotropic helicoid is right- or left-handed.

D. Experimental Data in the "Transition" Region

1. *Unbounded Fluids*

In addition to the existence of accurate data for spherical particles (M7b), a modest amount of recent experimental data are available for the resistance

[33] These properties follow from the fact that $\underline{\mathbf{F}}$ and $\underline{\mathbf{U}}$ are true vectors, whereas $\underline{\mathbf{T}}$ and $\underline{\boldsymbol{\omega}}$ are pseudovectors. And the entities appearing on opposite sides of an equality sign must be of the same tensorial type, either pseudo or true.

[34] The value $\underline{\overset{(t)}{\mathscr{K}}}{}^{[rt]} = -\frac{1}{2}\pi$, found by Rubinow and Keller [see Eq. (216)] for a sphere of radius a, has been confirmed by Cox (C17) by direct integration of Eq. (227b).

to translational motion of nonspherical bodies of various shapes in the so-called "transition" regime[35] (B2, C10, J2a, M4, M11, P6, W9). This region lies between the Stokes regime, where the drag coefficient C_D is asymptotically proportional to R^{-1}, and the Newton's law regime, where C_D is essentially independent of R. Based on the Brenner–Chester–Cox theory, Chow and Brenner (C8, C9) have proposed a theoretically sound method for the correlation of experimental data on nonspherical particles in the transition region. According to Eq. (234) the drag force F on symmetrical bodies at small, nonzero Reynolds numbers is related to the Stokes force F_0 by the general expression

$$F/F_0 = \text{function } (R^*) \tag{240}$$

where

$$R^* = \frac{|F_0|\rho}{6\pi\mu^2} \tag{241}$$

is a generalized Reynolds number. F_0 refers to the hypothetical Stokes drag force which the body would experience if it moved with the same speed U and orientation in a fluid of equal viscosity but zero density. The superfluous factor of 6π is introduced to make R^* identical with the actual particle Reynolds number, $aU\rho/\mu$, for a spherical particle of radius a.

The function appearing in (240) is a *universal* function, applicable to all particles possessing sufficient symmetry. Rather than employing the expression for this function given in Eq. (234), the functional relationship was established (C8, C9) by drawing a single, average curve through the experimental data of Malaika (M4, M11) for a wide variety of particle shapes setting in their stable orientations. The result is shown in Fig. 2.[36] For any particular orientation the Stokes force is necessarily of the form

$$|F_0|/6\pi\mu a U = C \tag{242}$$

where a is any representative particle length and C is a nondimensional constant, dependent only on particle shape and orientation. Since C is, in principle, determinable from a *single* terminal settling velocity experiment in

[35] In much of the engineering literature this is referred to as the "laminar" regime. Such terminology is inappropriate, since it incorrectly excludes the possibility that flow in the higher Reynolds number regime may also be laminar.

[36] A more sensitive test of the general theory would be furnished by a log-log plot of $(F/F_0) - 1$ vs R^*; however, the accuracy of existing data do not warrant such a critical test. [See the comments of Maxworthy (M7b) with regard to the accuracy of such a plot for spherical particles.]

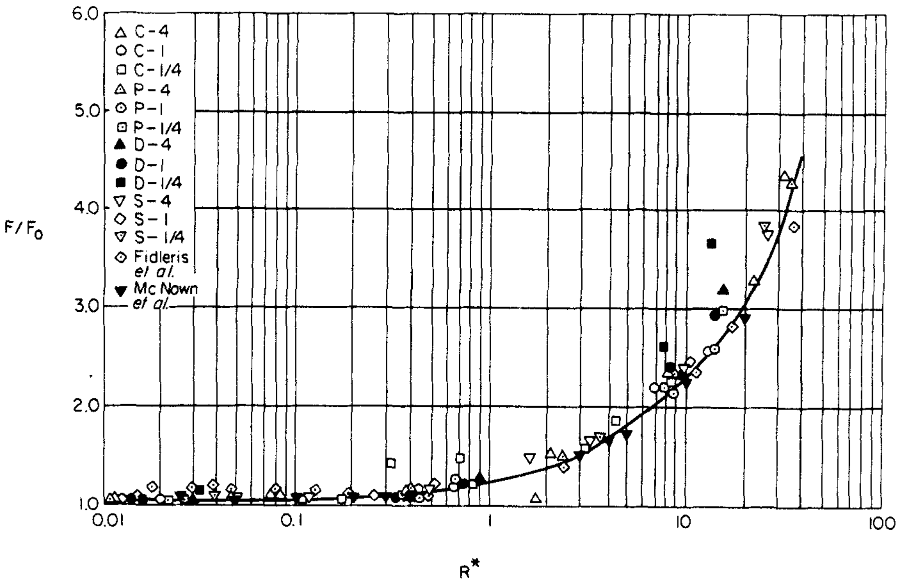

FIG. 2. Generalized drag coefficient vs generalized Reynolds number.

either the Stokes or transition regimes,[37] the plot, in essence, empirically characterizes the resistance of the body for a given orientation by means of a single parameter.

2. *Wall-Effects*

All other things being equal, wall-effects diminish with increasing particle Reynolds numbers (F6, M12, M13). Based on the analyses of Brenner (B15) and O'Brien (O1), Chow and Brenner (C8, C9) have suggested a universal plot encompassing both wall and inertial effects, providing that both effects are each not too large. For a symmetrical particle falling, say, along the axis of a circular cylinder of radius l, the dimensionless parameter

$$\xi^* = \frac{|F_0|}{6\pi\mu U l} \tag{243}$$

plays the role of a generalized particle-to-cylinder radius ratio (e.g., for a

[37] The C value can be derived from experimental data obtained in the transition region by cross-plotting Eq. (240) in the form

$$\frac{F}{F_0} = \text{function}\left(\frac{|F|\rho}{6\pi\mu^2}\right)$$

$|F|$ is then merely the weight of the settling particle corrected for the buoyancy of the fluid.

sphere of radius a, $\xi^* = a/l$). According to the underlying theory [38](B15, O1),

$$F/F_0 = \text{function}\ (R^*, \xi^*) \tag{244}$$

where the function appearing above is universally applicable to all sufficiently symmetrical particles. F_0 is, again, the hypothetical Stokes force on the particle in an unbounded fluid. For $R^* = 0$, Eq. (244) is equivalent to (135), with appropriate changes in notation. The *spherical* particle data of Fidleris and Whitmore (F5, F6) and McNown *et al.* (M10, M14) were used (C8) to prepare the required plot of Eq. (244). It seems likely that the theory is applicable for $R^* < 20$ and $\xi^* < 0.2$, but insufficient data are available to provide a truly satisfactory test.

E. Lateral Migration in Tubes

1. *Introduction*

Segré and Silberberg (S5, S6) observed that a single, rigid, neutrally buoyant, spherical particle suspended in a low Reynolds number Poiseuille flow migrates across the streamlines, its center ultimately attaining a stable equilibrium position at about 0.6 tube radius distant from the axis. This position is reached independently of the initial lateral location of the sphere relative to the tube axis; a neutrally buoyant sphere introduced near the wall migrates inward, whereas it migrates outward when introduced near the axis. This lateral migration effect was subsequently confirmed by others (G9a, J5, K5, K5b, O2, R4a, R4b) and found to be relatively insensitive to tube Reynolds number and sphere-cylinder diameter ratio. The observations of Segré and Silberberg spawned a remarkable number and variety of related lateral migration experiments. Especially important was the extension to non-neutrally buoyant spherical particles (D4b, E1, J5, R4a, R4b, T3a). Lateral forces were observed here too, but now—with the sole exception of the closely related two-dimensional Poiseuille flow experiments performed by Repetti and Leonard (R4a, R4b)—the particles no longer attained a fractionally eccentric equilibrium position. Rather, they migrated permanently to either the wall or the tube axis, depending essentially upon whether they were denser or lighter than the fluid and upon whether the Poisueille flow was up or down the tube. No sidewise (lift) forces can arise under comparable circumstances in the Stokes regime (B28, B30, S1). Such effects are therefore necessarily inertial in origin.

[38] The theory is most likely to be useful in the cross-plotted form

$$\frac{F}{F_0} = \text{function}\left(\frac{|F|\rho}{6\pi\mu^2}, \frac{|F|}{6\pi\mu U l}\right)$$

Radial migration phenomena are of obvious importance in the flow of suspensions through tubes, especially in dilute systems where they are not overshadowed by hydrodynamic interactions among particles. Thus, in the case of a dilute suspension of neutrally buoyant particles, the experiments of Segré and Silberberg (S5, S6) revealed a so-called "tubular pinch effect" in which the particles were observed to accumulate in a narrow annular portion of the tube centered at a distance of about 0.6 tube radius from the axis. At steady-state the particles are therefore nonuniformly distributed over the tube cross section, an effect previously noted by Tollert (T6) and implied in the work of others (M5a, M7a, S18, S19). Such anomolous behavior is important, for example, in interpreting suspension viscosity measurements obtained in capillary tubes (S7); for when the shear is nonuniform, as is true for Poiseuille flow, the contribution of each particle to the additional energy dissipation depends upon its radial position in the tube (B12). This results in an apparent non-Newtonian behavior (S7). Recognition of nonuniformities in radial distribution is also important in applying Archimedes' law to suspensions [see Eq. (145)] and in related applications dealing with the pressure drop through liquid fluidized beds and with the power requirements for pumping dilute slurries (H8). Experiments have shown (F8) that when deformable wood-pulp fibers suspended in water are pumped through a pipe a clear layer exists adjacent to the tube wall, though this manifestation of radial forces is as likely due to particle deformation as to fluid inertia (G9).

Unless the contrary is explicitly stated, the following discussion of experimental and theoretical results is restricted to single, rigid, spherical particles freely suspended in a Poiseuille flow within a circular tube of effectively infinite length. Notation is as follows: a = sphere radius; R_o = tube radius (l was used previously for this quantity); b = radial distance from tube axis to sphere center; $\beta = b/R_o$ = fractional distance from axis; b^* = stable equilibrium distance of sphere from tube axis; $\beta^* = b^*/R_o$; ρ = fluid density; ρ_p = particle density; μ = viscosity; $\nu = \mu/\rho$ = kinematic viscosity. All velocities defined below are measured relative to the fixed cylinder walls; $\mathbf{V}_m$ = mean velocity of flow vector (equal in magnitude to the superficial velocity and pointing parallel to tube axis in the direction of net flow); $\mathbf{U}$ = particle velocity vector—that is, the velocity of the sphere center; $\boldsymbol{\omega}$ = angular velocity of the sphere. The local velocity in the unperturbed Poiseuille flow is

$$\mathbf{u} = 2\mathbf{V}_m[1 - (R/R_o)^2] \tag{245}$$

where R is the distance from the axis to a general point in the fluid.

In resolving the various vector velocities into components it is convenient to define a right-handed system of Cartesian coordinates Ox_1, Ox_2, Ox_3 with origin O at the tube axis, as in Fig. 3. Ox_1 lies along the tube axis in the direction of net flow; Ox_3 lies parallel to the vector drawn perpendicularly from

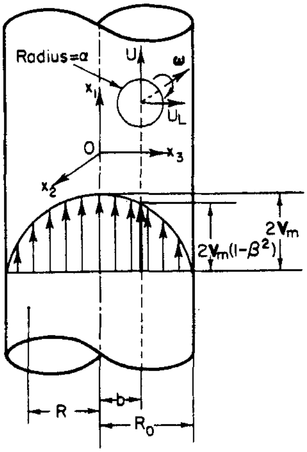

FIG. 3. Sphere in a Poiseuille flow.

the cylinder axis which passes through the sphere center. Corresponding unit vectors are denoted by $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. By definition we thus have $\mathbf{V}_m = \mathbf{i}_1 V_m$ where the scalar $V_m \geq 0$. The particle velocity may be expressed in the component form $\mathbf{U} = \mathbf{i}_1 U + \mathbf{i}_3 U_L$, where the axial velocity[39] U and radial velocity U_L may be positive or negative scalars. $U_L > 0$ implies migration toward the wall, whereas $U_L < 0$ implies migration towards the axis. Similarly, the hydrodynamic force exerted by the fluid on the particle may be expressed as $\mathbf{F} = \mathbf{i}_1 F_D + \mathbf{i}_3 F_L$. The "drag" force F_D and "lift" force F_L may also be positive or negative scalars.

Another quantity of importance is the axial slip velocity of the particle,

$$\mathscr{U}_p = U - 2V_m(1 - \beta^2) \tag{246}$$

which represents the particle velocity relative to the surrounding fluid. When the particle is neutrally buoyant and the Reynolds number small, the slip velocity may be obtained by setting $\mathbf{F} = \mathbf{0}$ in Eq. (142), yielding

$$\mathscr{U}_p = -\tfrac{4}{3} V_m \left(\frac{a}{R_o}\right)^2 + O\left(\frac{a}{R_o}\right)^3 \tag{247}$$

The negative sign is a consequence of the fact that the particle lags the fluid. When the particle is not neutrally buoyant it proves useful to introduce the settling velocity vector $\mathbf{U}_\infty$ of the particle in the *unbounded fluid at rest*. Of

[39] It would be more consistent with our general vector notation to reserve the symbol U for the magnitude of the vector $\mathbf{U}$. But since the lateral velocity is such a small fraction of the total velocity, the discrepancy between the magnitude of this vector and the absolute value of the axial velocity U is negligible.

particular interest is the axial component $U_\infty = \mathbf{i}_1 \cdot \mathbf{U}_\infty$ of this vector. At sufficiently small Reynolds numbers it is given by Stokes law,

$$U_\infty = \frac{2a^2(\rho_p - \rho)\mathbf{i}_1 \cdot \mathbf{g}}{9\mu} \tag{248}$$

and may be either positive or negative; $\mathbf{g}$ is the local acceleration of gravity vector, directed towards the center of Earth. If wall-effects are neglected in Eq. (142) the axial slip velocity is then simply

$$\mathscr{U}_p = U_\infty + \mathrm{O}(a/R_o) \tag{249}$$

which is obtained by noting that $\mathbf{F} = -6\pi\mu a\mathbf{U}_\infty$ in Eq. (142). When wall-effects are not negligible one finds for a vertical tube that

$$\mathscr{U}_p = U_\infty\left[1 - f(\beta)\frac{a}{R_o}\right] - \tfrac{4}{3}V_m\left(\frac{a}{R_o}\right)^2 + \mathrm{O}\left(\frac{a}{R_o}\right)^3 \tag{250}$$

The relative importance of inertial to viscous effects depends upon the following two Reynolds numbers (based on *diameter*):

$$\mathrm{Re}_t = 2R_oV_m/\nu = \text{tube Reynolds number} \tag{251}$$

$$\mathrm{Re}_p = 2a|\mathscr{U}_p|/\nu = \text{particle Reynolds number based on axial slip velocity} \tag{252}$$

The Reynolds number based upon the angular velocity of the particle is not an independent parameter for a freely supported particle; rather according to Eq. (89b) a homogeneous spherical particle will, in the absence of wall and particle inertia effects, rotate with the local angular velocity of the undisturbed flow at its center—at least at sufficiently small Reynolds numbers. This velocity is

$$\boldsymbol{\omega} = -\mathbf{i}_2\omega \tag{253a}$$

in which

$$\omega = 2bV_m/R_o^2 \tag{253b}$$

is the angular speed.

Most experimental data obtained to date have been interpreted on the basis of the Rubinow–Keller equation (216) for the lift force on a spinning, translating sphere in an unbounded fluid at rest (or in uniform flow) at infinity. The spin is assumed to be given by Eq. (253), whereas the velocity $\mathbf{U}$ appearing in Eq. (216) has usually been interpreted as the axial slip velocity $\mathbf{i}_1\mathscr{U}_p$. This gives rise to a lift force

$$F_L = \pi a^3\rho\omega\mathscr{U}_p[1 + \mathrm{O}(\mathrm{Re}_p)] \tag{254}$$

[where ω is given by Eq. (253b)] and hence, by Stokes law,

$$F_L = 6\pi\mu a U_L[1 + O(\text{Re}_p)] \tag{255}$$

to a radial migration velocity

$$\frac{U_L}{V_m} = \frac{1}{6}\left(\frac{2a\mathscr{U}_p}{\nu}\right)\left(\frac{a}{R_o}\right)\beta[1 + O(\text{Re}_p)] \tag{256}$$

Before discussing the credibility of this theory we shall summarize pertinent experimental data bearing on the phenomenon of radial migration at small Reynolds numbers. The range of variables encompassed by these experiments is tabulated in Table III. Unless otherwise stated, the experiments to be described were conducted in vertical tubes.

2. *Experimental Data*

a. Segré and Silberberg (S6). These experiments were performed with dilute suspensions of neutrally buoyant spheres, rather than with single particles. However, it appears that the suspensions were sufficiently dilute (0.33–4 particles/cm^3) to insure negligible hydrodynamic interactions among the particles. Particles were observed to attain stable equilibrium positions at about $\beta^* = 0.63$. Based primarily on observations made in the vicinity of the tube axis they correlated their measured radial migration velocities by means of the empirical relation

$$U_L/V_m = 0.34\,\text{Re}_t(a/R_o)^{2.84}\beta(\beta^* - \beta) \tag{257}$$

Further experiments were also conducted at tube Reynolds numbers up to 650, but the data scattered too much to conclusively determine whether or not the particles ultimately attained a stable equilibrium position.

b. Karnis, Goldsmith, and Mason (G7, G9, K5, K5b). An initial set of experiments by Goldsmith and Mason (G7, G9) was performed with essentially neutrally buoyant spheres for $\text{Re}_p \approx 10^{-6}$. No lateral migration was observed as the particle moved axially through distances as large as 3,500 cm. Very precise measurements were also made of the axial and angular velocities. For $a/R_o < 0.04$ they verified the theoretical relationships $U = 2V_m(1 - \beta^2)$ and $\omega = 2bV_m/R_o^2$ to within 1 percent. For $a/R_o = 0.1$ to 0.5, where wall-effects are not negligible, they observed significant departures from the no-slip axial velocity condition. According to Eqs. (246) and (247) one should have

$$\frac{2V_m(1 - \beta^2) - U}{2V_m(1 - \beta^2)}\,\frac{3}{2(a/R_o)^2} = \frac{1}{1 - \beta^2} \tag{258}$$

provided that $(a/R_o)(1 - \beta)^{-1} \ll 1$. Their data agree very well with this

TABLE III SUMMARY OF EXPERIMENTAL CONDITIONS FOR THE LATERAL

Observer	Fluid-particle density difference	Duct diameter, $2R_o$ (cm)	Sphere-to-cylinder radius, a/R_o
Segré and Silberberg (S6)	Dilute suspension of neutrally buoyant spheres	1.12	0.028–0.153
Goldsmith and Mason (G9)	Neutrally buoyant sphere	0.800	0.123–0.135
Karnis, Goldsmith, and Mason (K5)	Neutrally buoyant sphere	0.400	0.250
Oliver (O2)	Neutrally buoyant sphere (see also discussion for other cases)	0.940	0.245–0.361
Small and Eichhorn (E1)	More dense sphere in upflow in an inclined tube	1.07	0.146–0.301
Theodore and Brenner (T3a)	Immobile sphere in a horizontal tube	14.0	0.046–0.136
Denson, Christiansen, and Salt (D4b)	Less dense sphere in downflow	2.58	0.120–0.190
Day and Genetti (D3a)	More dense sphere in upflow	See Denson See Denson	See Denson See Denson
Repetti and Leonard (R4a)	Neutrally buoyant sphere	Rectangular duct 2.54×10.16 cm ($R_o \equiv 1.27$ cm)	0.187
	Less dense sphere in downflow		0.187
	More dense sphere in downflow		0.187
Jeffrey and Pearson (J5)	Neutrally buoyant sphere	3.25	0.046–0.089
	More dense sphere in downflow	3.25	0.046–0.089
	More dense sphere in upflow	3.25	0.046–0.089

[a] $\mathscr{U}_p$ was obtained from Eq. (246) using the experimentally measured values of U and V_m; when $\mathscr{U}_p$ was too small to be accurately determined from these experimental values, or when it was not experimentally determined, it was estimated from Eq. (250) using an average value of $f(\beta) = 2.1$ (see Table II), the latter number being quite accurate over the β range of all investigators.

[b] The U_∞ values required in this calculation were determined from the experimental particle radii, density differences, and viscosities by *assuming* the applicability of Stokes law,

$$|U_\infty| = \frac{2a^2|\Delta\rho|g}{9\mu}$$

Strictly speaking this is accurate only for $\mathrm{Re}_\infty < 0.5$. Thus, for those experiments for which $\mathrm{Re}_\infty > 0.5$ the U_∞ values represent hypothetical values, rather than the values that would be observed experimentally.

Tube Reynolds number, $Re_t = \frac{2R_o V_m \rho}{\mu}$	Particle Reynolds number[a], $Re_p = \frac{2a\lvert\mathcal{U}_p\rvert\rho}{\mu}$	Particle Reynolds number in an unbounded stationary fluid[b], $Re_\infty = \frac{2a\lvert U_\infty\rvert\rho}{\mu}$	Settling velocity–mean fluid velocity ratio[b], $\lvert U_\infty\rvert/V_m$	Eccentric equilibrium position of sphere center, $\beta^* = b^*/R_o$
2–30	< 0.01	?	0.0(?)	0.63
$0.80–1.6 \times 10^{-3}$	$2.2–5.3 \times 10^{-6}$[c]	$6.2–8.3 \times 10^{-7}$	0.0032–0.0077	No radial migration
≈ 0.01	$0.09–1.9 \times 10^{-2}$[c]	?	0.0 (?)	0.5
100–500	1.0–13.0	?	0.0(?)	0.47–0.63
860–2400	80–247[d]	306–1900[e]	1.7–21.8[e]	Lift force directed towards axis at all eccentric locations
2.3–42.8	0.13–8.1[d]	—	—	Lift force directed towards axis at all eccentric locations
208–890	6.0–120[f]	17–340	0.22–5.5	Axis
See Denson	< 120[f]	See Denson	See Denson	Axis
See Denson	550[f]	See Denson	See Denson	Undamped radial oscillation
500 (estimated)	?	0.0(?)	0.0(?)	?
500 (estimated)	27 (estimated)	35 (estimated)	0.38 (estimated)	Axis
500 (estimated)	10 (estimated)	24 (estimated)	0.26 (estimated)	Wall
11.2–76.8	0.010–0.11	0.0091–0.040	0.0025–0.040	0.68
22.7–116	0.37–1.3	0.28–1.7	0.028–0.43	Wall
22.0–180	0.28–1.6	0.28–1.7	0.029–0.42	Axis

[c] Because the direction of net flow oscillated periodically between upflow and downflow, the algebraic sign of U_∞ (relative to V_m) underwent similar changes. Accordingly, $\mathcal{U}_p$ was calculated from Eq. (247) rather than Eq. (250).

[d] Note that, since the sphere center did not move relative to the tube wall in these experiments, $U = 0$ and hence $\mathcal{U}_p = -2V_m(1-\beta^2)$.

[e] Note that the U_∞ values required in this calculation were obtained as outlined in footnote *b* to this table. No account was taken of the fact that $\mathbf{i}_1 \cdot \mathbf{g}$ in Eq. (248) is not the same as $g = 981$ cm/sec², due to the inclination of the tube.

[f] These particle Reynolds numbers are based on the *total* (axial plus radial) "slip velocity," defined by Denson (D4b) as

$$|\mathcal{U}_p| = \left[U_L^2 + \left\{ U - 2V_m(1-\beta^2) + \frac{4}{3} V_m \left(\frac{a}{R_o}\right)^2 \right\}^2 \right]^{1/2}$$

relationship for $\beta < 0.3$. Their failure to confirm Eq. (258) at the larger values of β is due to the fact that the condition $(a/R_o)(1-\beta)^{-1} \ll 1$ was violated, owing to the relatively large values of a/R_o employed. Measurements of the angular velocity of the sphere when its center lay within one diameter of the tube wall confirmed Vand's (V2, V3) theoretical relationship for the wall-effect due to a sphere rotating near a plane wall in Couette flow. This observed agreement should, however, be scrutinized more closely since Vand's analysis fails to satisfy the zero tangential-slip boundary condition at the wall. An exact solution of Vand's problem is given by Goldman, Cox, and Brenner (G5e). It is with this more rigorous result that the angular velocity data of Karnis *et al.* should be compared.

In a subsequent investigation in the range $\mathrm{Re}_p = 10^{-3}$ to 10^{-2}, Karnis, Goldsmith, and Mason (K5, K5b) observed both inward and outward radial migration, the equilibrium position occurring at approximately $\beta^* = 0.5$. Further experiments by Goldsmith and Mason (G9a, Fig. 15) confirmed these results.

That no radial migration was observed in the first set of experiments was undoubtedly due to the minuteness of the particle Reynolds numbers and, concomitantly, of the fact that despite the relatively enormous axial distances over which the radial position was observed, they were nevertheless insufficient to reveal the phenomenon. No matter how small the Reynolds number, inertial effects are present to some extent in all real flows. These effects, though practically infinitesimal at very small Reynolds numbers, are cumulative and would almost certainly have been observed had greater axial distances been involved. Commenting on this point, Rubinow (R6a) estimates that 4 to 8×10^3 cm would have been required in the original series of experiments. Goldsmith and Mason reply to Rubinow's remarks in Reference G9c.

c. Oliver (O2). Both inward migration from the wall and outward migration from the axis were observed for neutrally buoyant spheres. Equilibrium was attained at $\beta^* = 0.47$–0.63 (average $= 0.55$). Wall-effects were larger in these experiments than in those of other investigators, since the final positions of the sphere centers were within one diameter of the tube walls. Experiments were also conducted with neutrally buoyant spheres whose centers of mass were eccentrically located to prevent rotation. Here, again, both inward and outward migration were observed, though the final equilibrium position was much nearer the tube axis ($\beta^* = 0.07$–0.42; average $= 0.26$) than in the previous case.

In a few experiments conducted with nonneutrally buoyant spheres which were free to rotate, a sphere slightly more dense than the fluid in downflow attained equilibrium somewhat closer to the wall than previously. A less dense sphere in a downwardly flowing fluid drifted away from the wall.

d. Small and Eichhorn (E1, S13). In an ingeniously simple experiment, these authors indirectly measured both the drag and lift forces (as well as the angular velocity) for single, hydrodynamically supported spheres at various nonaxial positions. This was accomplished by suspending the particles in a less dense fluid flowing upward through an inclined tube. In this manner both the drag and lift forces were balanced by the corresponding components of the net gravitational force in the sphere. The radial force was directed inward, towards the tube axis, in all cases. They correlated their data on the lift force via the empirical equation

$$F_L = -4.4 \times 10^5 \rho v^2 (a/R_o)^4 (\beta/1 - \beta^2)^2 \tag{259}$$

but remark that only a tentative recommendation can be made for the formula. In this we concur, since it predicts a velocity-independent radial force. For comparison purposes this relation may be converted into an equivalent radial velocity by means of Eq. (255), though the use of Stokes law at their conditions of operation may not be warranted.

Equally tentative correlations are presented by them for the drag force and angular velocity.

e. Theodore and Brenner (T3a). Direct measurements were made of the lift force on a nonrotating sphere by means of a static-head opening drilled through the wall of a horizontally mounted pipe. This opening permitted a steel sphere to be connected via a thin vertical wire to an analytical balance, thereby furnishing a direct measurement of the lift. This force was always in towards the axis. Data were empirically correlated via the relation

$$F_L = -2.8 \times 10^2 \rho a^2 V_m{}^2 (a/R_o)[1 - 6.0(a/R_o)]\beta(1 - \beta^2) \tag{260}$$

Here again, by applying Eq. (255) this may be converted into an equivalent radial velocity.

A few additional experiments were conducted in which the sphere was allowed to rotate. Though not wholly conclusive, no statistically significant effect of particle rotation on lift was found.

f. Denson, Christiansen, and Salt (D4b). Experiments were performed with nonneutrally buoyant particles less dense than the entraining fluid. Density differences ranged from 3.3 to 10.4 percent of that of the fluid. Flow was vertically downward in all cases. An electromagnetic device was employed to release the particles into the fully developed portion of the flow, in contrast to the prior work of Segré and Silberberg (S6) where the particles entered with the entraining fluid. Data were obtained for the instantaneous radial, axial, and angular velocities of the spheres.

Particles always migrated permanently to the tube axis. At the lower particle Reynolds numbers (6.0–10.0) the approach to the axis was monotonic.

However, at the larger values (15.0–120) the sphere oscillated across the tube axis one or more times before the migration ceased—the frequency of the oscillation increasing with increasing Re_p and flow rate; that is, the radial motion was such that while remaining in the same vertical meridian plane the particle crossed the axis, continued to the other side of the tube (but not all the way to the wall), turned, and moved back towards the centerline, etc. The two distinct modes of motion observed in the different Re_p ranges are termed "overdamped" and "damped" by Denson.

Denson provides a quantitative theoretical analysis which adequately accounts for these phenomena. Agreement is quite good, especially in the range $\mathrm{Re}_p < 40$. No adjustable parameters appear in the treatment. The unsteady-state analysis depends critically upon the applicability of the Rubinow–Keller theory to the *instantaneous* particle motion, and the observed agreement is construed by Denson as evidence of its applicability for $\mathrm{Re}_p < 40$. (See, however, the remarks in the last paragraph of Section III,E,3.)

g. Day and Genetti (D3a). Denson (D4b) refers in his thesis to the related experimental work of Day and Genetti (D3a) conducted in collaboration with him at the University of Utah. Their studies revealed that a sphere, more dense than the entraining fluid, migrated to the tube axis when the fluid flowed vertically upward. These observations were made over the same range of particle and tube Reynolds numbers covered in Denson's investigation. At particle Reynolds numbers near 550, Day and Genetti found, however, that the oscillatory radial motion observed by Denson at the lower values of Re_p (up to 120) was no longer damped. Rather, the sphere oscillated across the tube axis with constant amplitude and frequency, in a manner similar to that of an undamped harmonic oscillator. They attributed this phenomenon to periodic shedding of vortices from the sphere, but Denson states that further studies would be required to substantiate this hypothesis.

h. Repetti and Leonard (R4a, R4b). These experiments, performed in a *rectangular* duct, revealed phenomena analogous to those observed in circular tubes. By using a gap width small in comparison with the breadth, an essentially two-dimensional Poiseuille flow was achieved. Drift was examined between the narrowly separated walls and was confined to the mid-plane between the other walls. A sphere more dense that the downflowing fluid migrated to the nearer duct wall. Less dense spheres in downflow migrated to the axis. When the fluid-particle density differences were very small (0.04 percent for the more dense sphere and 0.06 percent for the less dense one) the migration apparently ceased before the sphere reached its extreme position at the axis or wall.[40] This suggests, as pointed out by Repetti and

[40] The summary reported here is based upon only a preliminary account (R4a) of their work. A presumably more detailed account is now available (R4b).

Leonard, that the stable equilibrium position will depend upon the density difference.

i. Jeffrey and Pearson (J5). These represent perhaps the most complete and systematic radial migration experiments performed to date. Single spheres were introduced at various radial positions, ranging from the axis to the wall, in the fully developed portion of the flow. Their angular, axial, and radial velocities were measured by cinematographic techniques. Results were obtained for neutrally buoyant particles as well as for particles denser than the fluid in both upflow and downflow.

Neutrally bouyant particles. In the range $\beta = 0.0$–0.85 investigated, measured angular[41] and axial velocities were found to be only a few percent smaller than their theoretical values, $\omega = 2bV_m/R_o^2$ and $U = 2V_m(1 - \beta^2)$, respectively, the discrepancies being attributable to slight wall-effects. The effect of the lateral migration was to move the particles away from both the tube axis and wall, a particle eventually taking up a stable position at approximately $\beta^* = 0.68$. The empirical relationship selected to represent the radial migration velocity was[42]

$$U_L/V_m = 0.70(\mathrm{Re}_t)^{1/2}(a/R_o)^2\beta(\beta^* - \beta) \tag{261}$$

although Jeffrey points out that changing the coefficient to 0.10 and the exponent of a/R_o to 1.5 would have resulted in an equally satisfactory correlation of the data.

More dense particle in downflow. Density differences varied from 0.9 to 2.3 percent. In the range $\beta = 0.0$–0.85 investigated, the angular velocities agreed quite closely with Eq. (253). Axial velocity data were very well correlated by the empirical relation

$$U = (U_\infty + 2V_m)(1 - \beta^2) \tag{262}$$

This formula, however, is at odds with the theoretical formula

$$U = U_\infty + 2V_m(1 - \beta^2) \tag{263}$$

[41] Jeffrey remarks that Kohlman (K10a) also confirmed the correctness of Eq. (253) for $\mathrm{Re}_t(a/R_o)^2 < 0.25$. (This is a particle Reynolds number based on sphere radius and mean shear rate, $2V_m/R_o$, in the tube.)

[42] Jeffrey also reworked the original data of Segré and Silberberg (S6) and showed that their data could equally well be represented by the equation

$$U_L/V_m = 0.186(\mathrm{Re}_t)^{1/2}(a/R_o)^2\beta(\beta^* - \beta)$$

with roughly the same degree of precision as the original correlation, Eq. (257). The above relation differs by a factor of about 4 from Jeffrey's.

and is clearly in error in the limiting case $V_m = 0$. However, in the U_∞/V_m and β ranges investigated, the discrepancy is not too serious, particularly in view of the fact that Jeffrey did not explicitly take account of wall-effects [cf. Eqs. (246) and (250)]. When wall-effects are considered, the correct relation in Stokes flow is

$$U = U_\infty\left[1 - f(\beta)\frac{a}{R_o}\right] + 2V_m(1-\beta^2) - \tfrac{4}{3}V_m\left(\frac{a}{R_o}\right)^2 + O\left(\frac{a}{R_o}\right)^3 \tag{264}$$

The particles migrated permanently to the tube wall with a radial velocity given by

$$U_L/V_m = 0.044(\mathrm{Re}_\infty)^{1/2}\beta(1-\beta^2) \tag{265}$$

where

$$\mathrm{Re}_\infty = \frac{2a|U_\infty|}{\nu} \tag{266}$$

is the particle Reynolds number based on free-fall velocity in the unbounded fluid at rest.

More dense particle in upflow. Density differences varied from 0.9 to 2.3 percent; β values varied from 0.0 to 0.7. Measured angular velocities again agreed quite closely with the theoretical formula, Eq. (253). Axial particle velocity data scattered more than in the comparable downflow case, and Jeffrey was unable to decide between Eqs. (262) and (263). They both agree moderately well with the data though Jeffrey favors the latter in order to avoid the unacceptable inference that $U \to 0$ as $\beta \to 1$.

Particles migrated permanently to the tube axis with a radial velocity given by

$$U_L/V_m = -0.10(|U_\infty|/V_m)^{1/2}(\mathrm{Re}_\infty)^{3/4}\beta \tag{267}$$

Upon comparing Eqs. (265) and (267) with Eqs. (254)–(255) using the experimentally measured angular and axial slip velocities, Jeffrey concludes that the radial velocities predicted by the Rubinow–Keller theory are too small by an order of magnitude, except at the larger values of $|U_\infty|$, where their theory yields results too small by a factor of only about 1.5–3.

Jeffrey observed a curious "regimentation" phenomenon when a dilute suspension of particles more dense than the fluid flowed downward: ". . . The particles in the suspension migrated rapidly to the tube wall and there aligned themselves into regular vertical columns. Similarly, adjacent columns exerted an influence upon each other such that the columns moved parallel to one another in a vertical direction without disturbing the regimentation within a column until the centroids of any two adjacent particles in one column formed an equilateral triangle with the centroid of the nearest particle in the adjacent column. . . ." Regimentation phenomena of this type seem also to have been observed by Segré and Silberberg (S6, S4a). Jeffrey goes on to state that "The obvious physical explanation of this is that, under these conditions of flow, the wake of a particle considerably influences the motion of any particle close to it. It would therefore seem probable that the motion of a particle in a suspension would be different [from] that of the motion of an individual particle."

3. *Theoretical Interpretation*

As observed repeatedly, most investigators concerned with the radial migration problem have attempted to analyze their data on the basis of a rather broad interpretation of the Rubinow–Keller equation. In the neutrally buoyant case, i.e., $|U_\infty|/V_m \to 0$, this interpretation predicts [cf. Eqs. (256) and (247)] a radial velocity

$$U_L/V_m = -\tfrac{2}{9}\mathrm{Re}_t(a/R_o)^4\beta$$

which acts to move the particle inward, all the way to the tube axis. As recognized by Rubinow and Keller (R7) themselves, this conclusion is at odds with the experimental fact that the actual stable equilibrium position occurs at $\beta^* \approx 0.5$–0.68. Rubinow and Keller empirically suggest multiplying the above by the factor $-(\beta^* - \beta)/\beta^*$ to bring it at least into qualitative agreement with experiment, but this suggestion has little rational basis to commend it.

In the limiting case of a nonneutrally buoyant particle, where $|U_\infty|/V_m \to \infty$, (but $V_m \neq 0$), the slip velocity is given approximately by Eq. (249), so that Eq. (256) predicts

$$\frac{U_L}{V_m} = \frac{1}{6}\left(\frac{2aU_\infty}{\nu}\right)\left(\frac{a}{R_o}\right)\beta \tag{268}$$

According to this relation the particle migrates permanently to either the tube axis or wall, according as U_∞ is negative or positive. Stated in more general terms, the particle will attain a stable equilibrium position at the axis if the scalar $(\rho_p - \rho)\mathbf{V}_m\cdot\mathbf{g}$ is negative, and, conversely, will attain a stable equilibrium position at the wall if this scalar is positive. Thus, if flow is down the tube ($\mathbf{V}_m\cdot\mathbf{g} > 0$) the particle migrates to the axis if it is lighter than the fluid [cf. Denson (D4b), Oliver (O2), and Repetti and Leonard (R4a)], and to the wall if it is denser than the fluid [cf. Jeffrey (J5), Oliver (O2), and Repetti and Leonard (R4a)]. Similarly, in upflow ($\mathbf{V}_m\cdot\mathbf{g} < 0$) migration to the tube axis occurs if the particle is denser than the fluid [cf. Jeffrey (J5) and Day and Genetti (D3a)], and to the tube wall if it is lighter than the fluid. The experiments of Eichhorn and Small (E1) conducted with upflow in an inclined tube correspond to the former case ($\mathbf{V}_m \cdot \mathbf{g} < 0$; $\rho_p - \rho > 0$). Their observation that the lift force is directed inward towards the axis thus accords with the general criterion. Without exception, the data of every investigator of nonneutrally buoyant particles is therefore consistent with the conclusion that when the sphere lags the fluid it ends up at the axis, and when it leads the fluid it ends up at the wall—provided that the ratio $|U_\infty|/V_m$ is not too near zero. Though the Rubinow–Keller theory predicts this behavior, such agreement does not necessarily imply the applicability of their theory to the nonneutrally buoyant case.

In an attempt to explain their observed dependence of β^* on small density differences, Repetti and Leonard (R4a) employ the Rubinow–Keller relation (256). However, the required axial slip velocity is not obtained from Eq. (250), but rather from an empirical modification of it derived, in part, from the experimental axial slip velocity data of Goldsmith and Mason (G9). The equation they propose contains an adjustable parameter, which they evaluate by a heuristic argument. Though their theory yields a value of $\beta^* = 0.53$ for a neutrally buoyant particle and "explains" at least qualitatively the various modes of behavior observed for nonneutrally buoyant particles—including the extreme cases of migration to the wall or axis for large density differences, and the dependence of β^* on small density differences for *almost* neutrally buoyant particles—their proposal cannot be quantitatively correct; for it ignores the fact that for small a/R_o the slip velocity is correctly given by Eq. (250). And the term involving $f(\beta)(a/R_o)$, ignored by them, is an order of magnitude larger than the term of $O(a/R_o)^2$ which they retain, at least when $|U_\infty|$ and V_m are comparable in magnitude. But even apart from this, the Rubinow–Keller theory itself is not applicable to tube flow, as shown from different points of view by Saffman (S1a) and Cox and Brenner (C18). Their theories are discussed later. In particular, the latter advance an alternative, theoretically sound explanation of the phenomena observed by Repetti and Leonard.

The data of Theodore (T3a) for the lift force on a nonrotating sphere in a Poiseuille flow, and, to a lesser extent, the comparable data of Oliver (O2), show that lateral forces arise at small Reynolds numbers even in the absence of particle rotation. Thus, inertial lift forces can arise from "slip-shear" as well as from "slip-spin." That the character of these two forces is very different is shown clearly by the theoretical analysis of Saffman (S1a).

Saffman (S1a) analyzes the motion of a rigid spherical particle relative to an unbounded, uniform, simple shear flow, the translational velocity of the sphere center lying parallel to the streamlines of the undisturbed motion. Particle Reynolds numbers are assumed to be small, leading to a first-order analysis of inertial effects.[43] Let O' be an origin at the sphere center and denote by x_1', x_2', x_3' a right-handed system of Cartesian coordinates measured relative to O'. Referred to axes translating with the sphere, the boundary conditions considered by Saffman are

$$\mathbf{v}' = -\mathbf{i}_1(Sx_3' + \mathscr{U}_p) \qquad \text{at infinity} \tag{269a}$$

$$\mathbf{v}' = -(\mathbf{i}_2\omega) \times \mathbf{r}' \qquad \text{at } r = a \tag{269b}$$

where $\mathbf{r}' = \mathbf{i}_1 x_1' + \mathbf{i}_2 x_2' + \mathbf{i}_3 x_3'$. The translational velocity $\mathscr{U}_p$ of the sphere center relative to the surrounding fluid may be either positive or negative. $\mathscr{U}_p < 0$ signifies that the particle lags the fluid whereas $\mathscr{U}_p > 0$ indicates that it leads the fluid. No loss in generality results from taking the constant "velocity gradient" $S \geqslant 0$. The angular velocity of the sphere, $\boldsymbol{\omega} = -\mathbf{i}_2\omega$, may be arbitrarily specified.[44]

[43] The problem is solved via singular perturbation techniques requiring the formation of inner and outer expansions of the flow field. Using a Fourier transform in the manner originally suggested by Childress (C7a) in the solution of a similar problem, Saffman is able to avoid the very tedious computation of the outer flow field; rather, the calculation is brought to fruition solely from a knowledge of its Fourier transform. This general technique almost certainly has important computational advantages in a variety of related problems.

[44] Insofar as possible we have chosen the notation here to coincide with that of the tube flow problem. In the latter case, when the sphere is restrained to move parallel to the tube axis, the boundary conditions at infinity and on the sphere, measured relative to an origin which translates with the sphere, are: at infinity,

$$\begin{aligned}\mathbf{v}' = \mathbf{v} - \mathbf{U} &= \mathbf{i}_1 2V_m(1 - R^2/R_o^2) - \mathbf{i}_1 U \\ &= -\mathbf{i}_1[2V_m(R^2 - b^2)/R_o^2 + \mathscr{U}_p]\end{aligned}$$

and, on the sphere [see Eq. (253)],

$$\mathbf{v}' = -(\mathbf{i}_2\omega) \times \mathbf{r}'$$

Here, $\mathscr{U}_p$ is defined by Eq. (246) in the tube flow case. Compare these with Eqs. (269a) and (269b), respectively. Observe that as we move in the positive 3-direction (increasing R in the tube flow case and increasing x_3' in the shear flow case) the undisturbed velocity $\mathbf{v}'$ at infinity decreases algebraically in both cases. The quantities

$$Sx_3' \qquad \text{and} \qquad 2V_m(R^2 - b^2)/R_o^2$$

are comparable. They both vanish at the sphere center ($x_3' = 0$ and $R = b$, respectively). In comparing these terms, one should bear in mind that $x_3 = x_3' + b$ and $x_2 = x_2'$, where x_2, x_3 are measured from the cylinder axis. Furthermore, since $R^2 = x_2^2 + x_3^2$, we have the identity

$$2V_m\left(\frac{R^2 - b^2}{R_o^2}\right) = 2V_m\frac{(x_2')^2 + (x_3')^2 + 2bx_3'}{R_o^2}$$

In the vicinity of the sphere, $x_2' \to 0$ and $x_3' \to 0$, this approaches $4V_m b x_3'/R_o^2$. Comparison with the comparable term Sx_3' shows that the "equivalent" local shear rate for tube flow is

$$S = 4V_m b/R_o^2$$

Three independent Reynolds numbers arise when the equations of motion and boundary conditions are cast in nondimensional form: the particle Reynolds number based on slip velocity, $\mathrm{Re}_p = 2a|\mathscr{U}_p|/\nu$; a shear Reynolds number, $\mathrm{Re}_s = (2a)^2 S/\nu$; and a rotational Reynolds number, $\mathrm{Re}_\omega = (2a)^2|\omega|/\nu$. It is assumed that

$$\mathrm{Re}_p,\ \mathrm{Re}_s,\ \mathrm{Re}_\omega \ll 1 \tag{270}$$

Employing singular perturbation methods to solve the Navier–Stokes equations, Saffman obtains, for the drag, torque about the sphere center, and lift force,

$$\mathbf{F}_D = -\mathbf{i}_1 6\pi\mu a\mathscr{U}_p + \mathrm{O}\{\mu a^2\mathscr{U}_p(S/\nu)^{1/2}\} \tag{271}$$

$$\mathbf{T} = -\mathbf{i}_2 8\pi\mu a^3(\tfrac{1}{2}S - \omega) + \mathrm{o}(\mu) \tag{272}$$

$$\mathbf{F}_L = \mathbf{i}_3 81.2\mu a^2\mathscr{U}_p(S/\nu)^{1/2} + \mathrm{o}(\nu^{-1/2}) \tag{273}$$ [45]

These expressions are only valid when

$$\mathrm{Re}_s \gg (\mathrm{Re}_p)^2 \tag{274}$$

it also apparently being assumed that Re_ω is of the same or smaller order than Re_s.

To a first approximation the lateral migration velocity may be obtained by adding to Eq. (273) the Stokes law hydrodynamic force, $-\mathbf{i}_3 6\pi\mu a U_L$, and equating the total lift force to zero. This yields a lateral velocity

$$U_L = \frac{81.2}{6\pi}\mathscr{U}_p\left(\frac{a^2 S}{\nu}\right)^{1/2} \tag{275}$$

Since the couple vanishes for an unrestrained particle, one finds that $\omega = \frac{1}{2}S$ to terms of lowest order. To this order the sphere therefore spins with the

[45] This result may be compared with that found by Bretherton (B29a, Eq. 10) for the lift force per unit length experienced by a circular cylinder of radius a for the comparable two-dimensional problem:

$$\text{lift force/length} = \mathbf{i}_3 74.5\mu\mathscr{U}_p\left(\ln\frac{Sa^2}{\nu}\right)^{-2} + \mathrm{O}\left(\ln\frac{Sa^2}{\nu}\right)^{-3}$$

This force acts in the same direction as that found for the sphere. Note that there is a misprint in Bretherton's equation (10), a minus sign having been omitted before the second term.

Zierep (Z2), using the Euler equations, obtained the following approximate solution of Saffman's problem for an *inviscid* fluid:

$$\mathbf{F}_L = \mathbf{i}_3\frac{16\pi}{3}\rho a^3\mathscr{U}_p S$$

Rotation of the sphere was, however, not considered.

angular velocity of the unperturbed flow. For these same circumstances, the corresponding prediction of the Rubinow–Keller theory, obtained from Eqs. (254) and (255) by setting $\omega = \frac{1}{2}S$, is

$$U_L = \frac{1}{12}\,\mathscr{U}_p\left(\frac{a^2 S}{\nu}\right) \tag{276}$$

This differs radically from Eq. (275) although it does predict the correct direction of the lateral motion in the two possible cases, $\mathscr{U}_p > 0$ and $\mathscr{U}_p < 0$. Even apart from this formal difference is the fact that, to terms of lowest order, the lift force predicted by Saffman is independent of the angular velocity of the particle[46]; that is, the lift force would be the same even if the particle were prevented from rotating. On the other hand, the Rubinow–Keller lift force depends critically upon the particle rotation.

When the conditions required by the inequality (274) are met, the slip-shear Saffman lift force is larger, by an order of magnitude, than the slip-spin Rubinow–Keller lift force; for from Eqs. (275) and (276) we find

$$\frac{(U_L)_{\text{Saffman}}}{(U_L)_{\text{Rubinow–Keller}}} = O\left\{\frac{1}{(\mathrm{Re}_s)^{1/2}}\right\}$$

This ratio clearly becomes infinite in the limit $\mathrm{Re}_s \to 0$. Saffman demonstrates that terms of the Rubinow–Keller type, among others, will appear in the next higher-order approximation to Eq. (273).

Saffman's decision to analyze only the case for which the inequality (274) is satisfied rests entirely on the enormous analytical difficulties posed by the other possible limiting cases, $\mathrm{Re}_s \ll (\mathrm{Re}_p)^2$ and $\mathrm{Re}_s = O(\mathrm{Re}_p)^2$—the various Reynolds numbers still being supposed small compared with unity. Only in Saffman's case do single inner and outer expansions suffice; the other cases appear to require three or more distinct regions of expansion [cf. Chang (C5) and Shi (S9a) for examples of this in other contexts]. Thus, in a sense, the a priori restriction to the range encompassed by Eq. (274) rigs the problem at the outset to one in which the slip-shear force dominates that due to slip-spin. Consideration of the opposite case, $(\mathrm{Re}_p)^2 \gg \mathrm{Re}_s$, might have reversed the conclusion or, more likely, made the two lift forces of comparable order. Therefore, though Saffman's calculation unequivocally demonstrates the lack of universal applicability of the Rubinow–Keller theory to the lateral migration problem, it does not, without further argument, demonstrate their theory to be substantially in error in any plausible physical situation. Such arguments are, however, furnished by the following line of reasoning: In the case of tube flow the local shear rate S is of the order $4V_m b/R_o^2$. Hence, in the

[46] Note that in the derivation of Eq. (273) no use has been made of the relation $\omega = \frac{1}{2}S$; rather, the angular velocity was carried along as an arbitrary parameter.

neutrally buoyant case, where $\mathscr{U}_p$ is of the order $(4/3)V_m(a/R_o)^2$ [cf. Eq. (247)], one finds that $\mathrm{Re}_s/\mathrm{Re}_p = O(\beta)$, which is of order unity. This makes

$$\mathrm{Re}_s/(\mathrm{Re}_p)^2 = O(1/\mathrm{Re}_p)$$

As $\mathrm{Re}_p \to 0$ this ratio becomes infinite, showing that the restriction required by Eq. (274) is indeed met. Thus, at least in the neutrally buoyant case, the Rubinow–Keller theory is wholly inapplicable to Poiseuille flows. It should not be surmised, however, that Saffman's results themselves have any direct application to such flows; for Saffman's calculations take no account of either the nonconstancy of the local shear rate or of the presence of boundaries constraining the flow. And either of these effects may result in the appearance of contributions to the lift force more dominant than Saffman's, at least for some ranges of the many independent variables.

Saffman attempts to make plausible the applicability of his relationship to tube flows under certain conditions. Even granting the correctness of these arguments, the possible range of applicability is very limited indeed. To its credit, however, is the fact that it predicts certain qualitative features which agree with experiment. Thus, it furnishes the correct direction of the lift force observed by Theodore (T3a) for an immobile, nonrotating sphere in a Poiseuille flow. Furthermore, it indicates that the force on the sphere ought not to be significantly affected by allowing it to freely rotate. This too agrees with Theodóre's observations. Despite this agreement as regards gross behavior, Eq. (273) with $S = 4V_m b/R_o^2$ and $\mathscr{U}_p = -2V_m(1-\beta^2)$ fails to agree with Theodore's experimental results for the lift force, Eq. (260). The qualitative agreement may therefore be fortuitous.

In a direct attack upon the radial migration problem in essentially its full generality, Cox and Brenner (C18) succeeded in obtaining a first-order solution of the Navier–Stokes and continuity equations for the motion of a rigid spherical particle immersed in a Poiseuille flow within a circular tube of finite radius.[47] No couple acts on the particle, so it is free to rotate. It is presumed in the analysis that the sphere center moves parallel to the tube axis. The lateral force required to maintain the sphere at a fixed distance from the axis is computed and converted into an equivalent radial migration velocity by application of Stokes law to this sidewise motion.

When the equations of motion and boundary conditions are expressed in nondimensional form, the following functional relationship ultimately emerges:

$$\frac{U_L}{V_m} = \text{function}\left(\mathrm{Re}_t, \frac{a}{R_o}, \frac{U_\infty}{V_m}, \beta\right) \tag{277}$$

[47] Their solution actually applies to particles and ducts of more general shape, but only the results for the sphere and circular tube are cited here. They also give results for the case where the sphere is prevented from rotating.

The angular velocity of the particle is not an independent parameter. In fact, to the order of the approximation, the particle spins with the local angular velocity of the undisturbed fluid. The analysis assumes that

$$\mathrm{Re}_t \ll 1 \tag{278a}$$

$$a/R_o \ll 1 \tag{278b}$$

Using singular perturbation methods[48] a solution is obtained in the form of a double expansion in the two parameters Re_t and a/R_o. No restriction is placed on the parameter β except that the particle not be too near the wall; i.e., $a/(R_o - b) \ll 1$, or, what is equivalent,

$$\frac{a}{R_o}\left(\frac{1}{1-\beta}\right) \ll 1 \tag{279}$$

This constitutes a very mild restriction. The parameter $|U_\infty|/V_m$ may be large or small.

As might be expected on the basis of prior observations, the form of the solution depends, at least in part, on the order of $|U_\infty|/V_m$ relative to that of a/R_o. Five separate cases arise, ranging from that for a neutrally buoyant particle to that for a particle settling in an otherwise quiescent (but bounded) fluid. The results are tabulated below.

(i) Neutrally buoyant sphere: $|U_\infty|/V_m \ll (a/R_o)^2$;

$$U_L/V_m = \mathrm{Re}_t(a/R_o)^3\, F_{\mathrm{I}}(\beta) \tag{280}$$

(ii) Intermediate case between a neutrally buoyant and nonneutrally buoyant sphere: $|U_\infty|/V_m = O(a/R_o)^2$;

$$U_L/V_m = \mathrm{Re}_t(a/R_o)[(a/R_o)^2\, F_{\mathrm{I}}(\beta) + (U_\infty/V_m)\, F_{\mathrm{II}}(\beta)] \tag{281}$$

(iii) Nonneutrally buoyant sphere: $(a/R_o)^2 \ll |U_\infty|/V_m \ll 1$;

$$U_L/V_m = \mathrm{Re}_t(a/R_o)(U_\infty/V_m)\, F_{\mathrm{II}}(\beta) \tag{282}$$

[48] In a sense, the case where the flow is bounded is conceptually easier to treat than is the comparable unbounded flow; for when the flow is bounded the disturbance created by the sphere dies away exponentially with axial distance (T1a, S16) rather than inversely with some power of the distance, as is true for the unbounded case. For this reason, the outer boundary conditions satisfied by the lower-order terms of the inner expansion can be determined without having to apply the "matching principle" except in a trivial sense; that is, the outer boundary conditions satisfied by the lower-order terms of the inner expansion are identical to the physical outer boundary conditions. Since the force and torque on the sphere can be determined from the inner expansion alone, it is therefore not necessary to compute *any* terms in the outer expansion in order to obtain the leading terms in the force and torque. Thus, the analysis differs greatly from those of Rubinow and Keller (R7) and Saffman (S1a).

(iv) Intermediate case between a nonneutrally buoyant sphere and no net flow in the tube: $|U_\infty|/V_m = O(1)$;

$$U_L/V_m = \mathrm{Re}_t(a/R_o)(U_\infty/V_m)[F_{\mathrm{II}}(\beta) + (U_\infty/V_m)\,F_{\mathrm{III}}(\beta)] \tag{283}$$

(v) No net flow in tube: $1 \ll |U_\infty|/V_m$;

$$U_L/U_\infty = (2aU_\infty/\nu)\,F_{\mathrm{III}}(\beta) \tag{284}$$

provided that the following additional restriction,

$$\mathrm{Re}_\infty = \mathrm{Re}_t(a/R_o)|U_\infty|/V_m = 2a|U_\infty|/\nu \ll 1$$

is met in Case (v).

Note that Case (ii) is merely a composite of Cases (i) and (iii), while Case (iv) is a composite of Cases (iii) and (v).

The three nondimensional functions of β appearing in these expressions are defined in terms of integrals involving the fundamental solution (L1) of Stokes equations for an eccentrically located vector point force of arbitrary orientation in an infinitely long circular cylinder for the case where the fluid is at rest at infinity. This fundamental solution is defined explicitly in the following way. All distances appearing in the following are made *dimensionless with the cylinder radius*. Let $\underline{\mathbf{r}}$ denote the nondimensional position vector of the singular point force relative to an arbitrary origin and let $\underline{\bar{\mathbf{r}}}$ be the nondimensional position vector of any other point in the fluid relative to this same origin. The fundamental solution required is the dyadic "velocity" field, $\mathbf{V} = \mathbf{V}(\underline{\bar{\mathbf{r}}}, \underline{\mathbf{r}})$, and vector "pressure" field, $\mathbf{P} = \mathbf{P}(\underline{\bar{\mathbf{r}}}, \underline{\mathbf{r}})$, satisfying the inhomogeneous[49] Stokes equations

$$\underline{\bar{\nabla}}^2\mathbf{V} - \underline{\bar{\nabla}}\mathbf{P} = -\mathbf{I}\,\delta(\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}}) \tag{285a}$$

$$\underline{\bar{\nabla}} \cdot \mathbf{V} = \mathbf{0} \tag{285b}$$

and boundary conditions

$$\mathbf{V} = \mathbf{0} \quad \text{on the cylinder wall} \tag{286}$$

and

$$\mathbf{V} \to \mathbf{0} \quad \text{at infinite axial distances on either side of the singularity} \tag{287}$$

[49] Alternatively, by adding the supplementary condition

$$\mathbf{V} \to \frac{1}{8\pi|\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}}|}\left\{\mathbf{I} + \frac{(\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}})(\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}})}{|\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}}|^3}\right\} \quad \text{as} \quad (\underline{\bar{\mathbf{r}}} - \underline{\mathbf{r}}) \to \mathbf{0}$$

one may solve instead the corresponding *homogeneous* Stokes equations.

Note that though V and P defined by Eqs. (285)–(287) are dimensionless, in order to keep the notation concise they have not been underlined, despite our usual practice for such entities.

Here, δ denotes the Dirac delta function, and **I** the idemfactor [cf. Eqs. (193)–(194)]. Physically, the Cartesian tensor equivalent of **V**, namely, V_{ij}, represents the ith component of the Stokes velocity field at $\bar{\underline{\mathbf{r}}}$ due to a unit point force exerted on the fluid at $\underline{\mathbf{r}}$ in the jth direction.

In order to express the various $F(\beta)$ in terms of **V**, let $\underline{\mathbf{r}} \equiv (\underline{x}_1, \underline{x}_2, \underline{x}_3)$ be Cartesian coordinates having the sense previously assigned to the 1, 2, and 3 directions (see Fig. 3). (It is not essential, however, that the origin lie along the cylinder axis.) Furthermore, let $\underline{R}$ and $\bar{\underline{R}}$ be radial cylindrical distances measured from the axis to the points $\underline{\mathbf{r}}$ and $\bar{\underline{\mathbf{r}}}$, respectively. The functions in question are then given by the expressions

$$F_{\mathrm{I}}(\underline{R}) = \frac{40\pi}{3}\underline{R}\sum_{k=1}^{3}\int_{\underline{\bar{\Gamma}}}\left[(\underline{R}^2 - \bar{\underline{R}}^2)V_{k3}\frac{\partial}{\partial\bar{\underline{x}}_1}\left(\frac{\partial V_{k1}}{\partial\underline{x}_3} + \frac{\partial V_{k3}}{\partial\underline{x}_1}\right)\right.$$
$$\left. + \left\{\frac{\partial}{\partial\bar{\underline{x}}_k}(\underline{R}^2 - \bar{\underline{R}}^2)\right\}V_{13}\left(\frac{\partial V_{k1}}{\partial\underline{x}_3} + \frac{\partial V_{k3}}{\partial\underline{x}_1}\right)\right] d\bar{\underline{\Gamma}} \tag{288}$$

$$F_{\mathrm{II}}(\underline{R}) = 6\pi\sum_{k=1}^{3}\int_{\underline{\bar{\Gamma}}}\left[(\underline{R}^2 - \bar{\underline{R}}^2)V_{k3}\frac{\partial V_{k1}}{\partial\bar{\underline{x}}_1} + \left\{\frac{\partial}{\partial\bar{\underline{x}}_k}(\underline{R}^2 - \bar{\underline{R}}^2)\right\}V_{13}V_{k1}\right] d\bar{\underline{\Gamma}} \tag{289}$$

$$F_{\mathrm{III}}(R) = -3\pi\sum_{k=1}^{3}\int_{\underline{\bar{\Gamma}}}V_{k3}\frac{\partial V_{k1}}{\partial\bar{\underline{x}}_1}\, d\bar{\underline{\Gamma}} \tag{290}$$

in which $d\bar{\underline{\Gamma}} \equiv d\bar{\underline{x}}_1\, d\bar{\underline{x}}_2\, d\bar{\underline{x}}_3$ is a volume element, and $\bar{\Gamma}$ denotes integration over the entire space (including the singularity) lying within the cylinder.

Calculation of the Lorentz dyadic **V** can be performed by an obvious generalization of the work of Brenner and Happel (B28). Though straightforward in principle, the algebraic effort required is prodigious, and the work is as yet incomplete. At this stage we are therefore unable to describe any general features of the $F(\beta)$. In order to explain existing experimental data these functions would have to possess at least the following properties:

$$F_{\mathrm{I}}(\beta) = \begin{cases} 0 & \text{at } \beta = 0 \\ >0 & \text{for } 0 < \beta < \beta^* \\ 0 & \text{for } \beta = \beta^* \approx 0.6 \\ <0 & \text{for } \beta^* < \beta < 1 \end{cases} \tag{291}$$

and

$$F_{\mathrm{II}}(\beta) = \begin{cases} 0 & \text{at } \beta = 0 \\ >0 & \text{for } 0 < \beta < 1 \end{cases} \tag{292}$$

and

$$F_{\mathrm{III}}(\beta) = \begin{cases} 0 & \text{for} \quad \beta = 0 \\ <0 & \text{for} \quad 0 < \beta < 1 \end{cases} \tag{293}$$[50]

Though quantitative comparisons with experimental data are denied us as yet, various qualitative comparisons are possible. Thus, except for a slight difference in the exponent of a/R_o, the functional form of Eq. (280) agrees with that observed by Segré and Silberberg, Eq. (257). According to Eq. (280) the equilibrium position attained by the sphere corresponds to $F_{\mathrm{I}}(\beta^*) = 0$. This implies that the equilibrium position of a neutrally buoyant sphere should be independent of Re_t and a/R_o, at least when these are sufficiently small. The experiments of Segré and Silberberg (S6), Oliver (O2), Karnis *et al.* (K5, K5b), and Jeffrey (J5), covering a rather large range of these variables, tends to bear this out. Equation (282) indicates that the direction of lateral migration depends upon the algebraic sign of U_∞/V_m in the nonneutrally buoyant case. This conclusion too is borne out by the experiments of all investigators.

In practical problems one is unlikely to encounter particles which are precisely neutrally buoyant. For this reason, Case (ii) is likely to be of greater interest in applications than is Case (i). According to Eq. (281) the stable equilibrium position depends upon the dimensionless ratio

$$\Lambda = \frac{U_\infty/V_m}{(a/R_o)^2} \tag{294}$$

which may be either positive or negative. According to the general properties noted in Eqs. (291) and (292), the equilibrium position β^* will lie nearer to the axis than 0.6 when the particle lags the fluid ($\Lambda < 0$) and nearer to the wall than 0.6 when the particle leads the fluid ($\Lambda > 0$). These phenomenon appear to have been observed by Repetti and Leonard (R4a) in a *two-dimensional* Poiseuille flow, but they do not report sufficient data in their initial communication to settle the matter unequivocally. Failure of previous investigators to observe this phenomenon in circular tubes is not surprising, since the range of density differences required to prevent migration of the particle all the way

[50] Oliver (O2) observed that a sphere settling slowly near the wall of a tube in an otherwise stagnant fluid moved inwardly, towards the tube axis. Karnis (K4a, K5b), in a series of more detailed measurements, made similar observations and followed the motion of the sphere all the way to the tube axis. It is on the basis of these observations that $F_{\mathrm{III}}(\beta)$ is concluded to be negative. This conclusion accords with Oseen's (O4) theoretical finding [also summarized in Berker (B4, p. 328)] that, when small inertial effects are considered, a sphere moving parallel to a plane wall in a semi-infinite fluid experiences a *repulsive* hydrodynamic force, due to the sourcelike behavior of the flow at points distant from the sphere not lying within the wake [cf. Lamb (L5, p. 613)].

to the tube axis or wall is remarkably narrow. Thus, Repetti and Leonard find that, starting with neutrally buoyant particles, small density differences resulting from temperature changes of the order of $\pm 5°C$ (corresponding to density differences of the order of ± 0.05 percent) are quite sufficient to radically alter the ultimate equilibrium position.

An order of magnitude estimate of the density differences required to observe the Repetti–Leonard phenomenon in a tube may be obtained by arbitrarily considering the Λ range, $|\Lambda| = (0.1\text{–}10.0)(4/3)$ [see Eq. (250) for the 4/3 factor]. Assuming the validity of Stokes law, $|U_\infty| = 2a^2|\Delta\rho|g/9\mu$, this requires that the fluid-particle density difference lie in the range

$$\frac{|\Delta\rho| g R_o^{\,2}}{\mu V_m} = 0.6\text{–}60 \tag{295}$$

or, in terms of the tube Reynolds number,

$$\frac{|\Delta\rho|}{\rho} = (0.3\text{–}30)\frac{\mathrm{Re}_t \nu^2}{g R_o^{\,3}} \tag{296}$$

For example, for water ($\nu \approx 10^{-5}$ ft^2/sec), taking $\mathrm{Re}_t = 1$ and $R_o = 1$ in., this yields $100 \times |\Delta\rho|/\rho \approx 10^{-7}\text{–}10^{-5}$ *percent*!!! Thus, the effect is never likely to be observed in water. On the other hand, at the same tube Reynolds number, increasing the viscosity by 100-fold and decreasing the radius by one-half multiplies the above percentages by a factor of about 10^5. Hence, observation of the phenomenon should not be too difficult with a sufficiently viscous fluid.

As mentioned previously, Repetti and Leonard (R4a) attempted an explanation of this phenomenon on the basis of a questionable modification of the Rubinow–Keller theory. It is now clear, however, that their proposal cannot be correct.

Since the equilibrium position for Case (ii) is independent of particle size and depends only on the parameter $\Delta\rho\, g R_o^{\,2}/\mu V_m$, a novel separation technique suggests itself. Particles of different density will accumulate in different streamlines, irrespective of particle size. By choosing the operating conditions carefully, it appears possible to separate particles whose densities differ only infinitesimally from one another.

Granting the correctness of the inequalities set forth in Eqs. (292) and (293), it appears from the results of Case (iv) that there may exist fractionally eccentric equilibrium positions originating from circumstances very different from those already investigated to date under the general domain of Cases (i) and (ii).

With regard to the interpretation of available experimental data, the only outstanding point seemingly in need of further discussion is the observation by Denson (D4b) that the Rubinow–Keller theory agrees well with his data. Equation (282) may be rewritten as

$$\frac{U_L}{V_m} = \frac{2aU_\infty}{\nu} F_{\text{II}}(\beta) \tag{297}$$

Comparison with Eq. (268), bearing in mind that Denson's data only covers a limited range of a/R_0 values, suggests that this agreement might be fortuitous if $F_{\text{II}}(\beta)$ were approximately equal to $(0.12\text{–}0.19)\beta/6$. Whether or not this is indeed the case awaits the numerical evaluation of the function in question.

4. *Radial Migration Under More General Circumstances*

The preceding discussion of experimental and theoretical studies pertaining to lateral migration applies only to the case of a rigid spherical particle suspended in a fully-developed, laminar, Newtonian flow within a circular tube. As discussed in subsequent paragraphs, related experimental and theoretical work on radial migration has also been conducted for a variety of situations in which one or more of these restrictions is relaxed.

Lateral migration is possible even in the absence of inertial effects if the particle is not spherical. Thus, on the basis of Stokes equations, Bretherton (B30) demonstrated theoretically that a rigid, neutrally buoyant, anisotropic particle lacking a center of symmetry can undergo lateral migration in Couette and Poiseuille flows. Similarly, Goldsmith and Mason (G7, G9) experimentally observed radial migration of *flexible* rods and disks to the axis of a Poiseuille flow. This occurred at values of $\text{Re}_p < 10^{-6}$ and $\text{Re}_t < 2 \times 10^{-2}$, where no discernible lateral motion was detected for rigid spheres (or for rigid rods or disks, either[51]).

Also in the context of Stokes flow, a neutrally buoyant *liquid droplet* immersed in a Poiseuille or other nonuniform shear flow deforms in response to the local shear and migrates across the streamlines, provided that the viscosity ratio, η = droplet viscosity/continuous phase viscosity, is not too large. When the latter condition is met, the droplet acquires the ellipsoidal shape described by Eq. (122a). In a Poiseuille flow this leads to the conclusion that the principal axes of a nonaxially situated droplet are *permanently* inclined at definite angles to the tube axis (though the extent of the deformation decreases as it approaches the tube axis), leading to an asymmetry of the flow and, hence, to the existence of radial forces arising from the nonuniformity of the shear. [In connection with this observation it should be noted that in contrast to the behavior of ellipsoid-like liquid droplets, *rigid* disks and rods do not maintain a fixed orientation relative to the tube walls, but rather undergo a periodic rotation (G9) of the type predicted by Jeffery (J4). Thus,

[51] At larger particle Reynolds numbers ($\text{Re}_p \approx 10^{-3}$), where inertial effects are more significant, rigid, neutrally buoyant rods and disks behave in essentially the same manner as do rigid spheres. They migrate inward if introduced near the wall and outward if introduced near the axis, attaining a stable terminal position at about $\beta^* = 0.5$ (K5, K5b).

averaging over one period, any radial forces associated with the asymmetric orientation of a *rigid* centrally symmetric particle relative to the streamlines will tend to cancel.] Such droplets always migrated rapidly to the tube axis (G9), at least for $\eta < 10$ (K5). This phenomenon was observed at conditions ($Re_p < 10^{-6}$) where comparable experiments with *rigid*, neutrally buoyant particles (including disks and rods) yielded no detectable radial motion. No radial motion was observed under similar conditions when the viscosity ratio was large, i.e., $\eta = 50$ (K5, T1). This anomolous behavior appears to be connected with the fact that, according to Eq. (122b), the droplet behaves like a rigid sphere for such large viscosity ratios, its deformation being nil. Moreover, at somewhat larger values of Re_p ($> 10^{-3}$), these highly viscous droplets attained stable equilibrium positions at about $\beta^* = 0.63$ (T1), in a manner identical to that observed for rigid spheres.

If the droplet is not neutrally buoyant it may migrate either to the axis or to the wall, depending, at least in part, upon the algebraic sign of the density difference and the direction of net flow relative to the gravity field. Thus, Jeffrey (J5) performed a few qualitative experiments with liquid droplets whose density was greater than that of the entraining liquid flowing in Poiseuille flow. The Reynolds number range was similar to that previously described in connection with his experiments on rigid spheres. With flow up the tube the droplets migrated permanently to the axis; with flow down the tube they migrated to the wall. Purely visual observations indicated radial migration velocities to be of the same order of magnitude as those observed for rigid spherical particles. The droplet migrations observed by Jeffrey were thus probably due, in large measure, to inertial effects. It should be noted, however, that these effects, including the observed sense of the radial motion, would be expected even in the Stokes regime; for, as has been pointed out, the droplet acquires the shape of an ellipsoid whose axes are permanently inclined relative to those of the tube. And even a *rigid* ellipsoid with its axes obliquely inclined to the direction of gravity will drift laterally as it settles in a quiescent fluid—the sense of the drift depending upon whether the particle is denser or lighter than the fluid.[52]

[52] A theoretical analysis of the Stokes flow problem for a nonneutrally buoyant droplet is clearly called for. Germane to this problem is the theoretical analysis of Haberman (H3), dealing with axially symmetric Stokes flow relative to a liquid droplet at the axis of a circular tube, and Taylor and Acrivos' (T2c) extension of the classical Hadamard–Rybczyński liquid droplet problem to the case of nonzero Reynolds numbers. In particular, Haberman shows that the assumption of a spherical shape for the droplet in a tube is incompatible with the differential equations and boundary conditions. Taylor and Acrivos (T2c) point out that, though Hadamard (H3a) and Rybczyński (R10) were able to solve the Stokes flow problem by assuming a spherical shape for a liquid droplet, irrespective of the magnitude of the interfacial tension, the correctness of their a priori assumption was, to some extent, fortuitous. These remarks are undoubtedly pertinent to the resolution of Haberman's "paradox" and, ultimately, to the solution of the nonaxially symmetric droplet problem.

The radial migration of a neutrally buoyant liquid droplet with its center lying in the mid-plane between two equal counter-rotating disks was theoretically studied by Chaffey, Brenner, and Mason (C2) on the basis of Stokes equations. This constitutes a simple example of a nonuniform shear field, the rate of shear increasing outwardly from the axis of rotation. The unperturbed velocity in the mid-plane between these disks is zero, so that radial motion arises entirely from the nonuniformity of the shear over the droplet surface. The sense of the migration is found to depend upon the ratio η of drop viscosity to continuous phase viscosity. For $\eta > 0.139$ the droplet migrates inwardly, towards the axis of rotation of the disks. For $\eta < 0.139$ the migration is outward. When $\eta = 0.139$ no migration occurs. In another paper (C3), these same authors also considered the lateral motion of a deformable liquid droplet near a plane wall when suspended in another liquid undergoing Couette flow. Migration away from the wall is predicted on the basis of Stokes equations.

Karnis, Goldsmith, and Mason (K5, K5d) performed experiments on the radial migration of rigid spheres suspended in a *viscoelastic* fluid flowing through a circular tube. Migration towards the axis was observed under conditions where inertial effects would normally be expected to be nil. The observed radial motion is thus apparently attributable to the non-Newtonian properties of the fluid. The unperturbed velocity profile is very flat over the central portion of the tube. Particles placed in this region neither rotated nor moved radially. Experiments are also reported for rods and disks (G9b, K5d).

In regard to *nonsteady* tube flows, Mason *et al.* have observed both inward and outward radial migration of rigid, neutrally buoyant spheres in oscillatory (S9b, G9b) and pulsatile (T1) flows in circular tubes at frequencies up to 3 cps, at which frequencies inertial effects are likely to be important.[53] We refer here to inertial effects arising from the *local* acceleration terms in the Navier-Stokes equations, rather than from the *convective* acceleration terms. In the oscillatory case the spheres ($a/R_o \approx 0.10$) attained equilibrium positions at about $\beta^* = 0.85$. Important Reynolds numbers here are those based upon mean tube velocity for one-half cycle and upon frequency. Nonneutrally buoyant spheres in oscillatory flow migrate permanently to the tube axis, irrespective of whether they are denser or lighter than the fluid (K4a).

Convective radial velocities arise during the entrance of a fluid into a tube. The particles in an initially uniform suspension will therefore be redistributed upon entering the tube, giving rise to anomolous effects (K5c, M5a, M7a, S18, S19).

Again, in relation to end effects, a statistical type of "tubular pinch effect"

[53] An oscillatory flow is one in which there is no *net* flow. A pulsatile flow is an oscillatory flow superposed on a steady net flow. The latter is important in blood flow phenomena, the pulsations arising from the periodic action of the heart.

is possible in batch, liquid-fluidized beds when the densities of particles and fluid are not matched (H8). This occurs even in the Stokes regime and has a simple explanation. The phenomenon is well known experimentally, and manifests itself by the existence of a very high concentration of particles near the tube walls.

Radial particle motions also arise during the initial filling of an empty tube with a suspension of neutrally buoyant particles. This "meniscus effect" (K5c) is most pronounced near the interface between the free surface of the liquid and the atmosphere. This type of end effect occurs even in the absence of inertial effects, i.e., in the Stokes regime.

Many of the single-particle radial migration experiments described above are currently being extended to concentrated suspensions of particles by Goldsmith, Mason, and their co-workers (K5a). This work is likely to prove of great significance to chemical engineers and others. The bulk of their work to date on all aspects of fluid-particle dynamics is critically surveyed in their review chapter on "The Microrheology of Dispersions" (G9d).

IV. Heat- and Mass-Transfer Analogies, Brownian Motion, and Summary

A. Heat- and Mass-Transfer Analogs

In view of the many analogies between momentum-transport and heat- and mass-transport, it is not surprising to find applications of much of the preceding hydrodynamic theory in the latter areas of scalar transport. It is not our intention here to systematically survey the general field of heat- and mass-transfer at small Reynolds numbers. Rather, we shall restrict ourselves exclusively to situations in which the Péclet number is *small*. For it is here that the analogies are strongest. Since the Péclet number is a global measure of the ratio of convective to molecular energy (or mass) transport it plays the same role in heat- and mass-transfer as does the Reynolds number in momentum transfer. Techniques for treating low Reynolds number heat- and mass-transfer at *large* Péclet numbers are admirably discussed in Levich's book (L10a).

With appropriate simplifying assumptions, the energy equation for steady-state forced-convection heat-transfer may be written as

$$\mathbf{v} \cdot \nabla T = \alpha \nabla^2 T \tag{298}$$

where T is the temperature and α the thermal diffusivity. Attention is confined to the case where $\mathbf{v}$ refers to a Stokes flow past the body. In this case the Nusselt number is a function only of the Péclet number, since the Reynolds number is no longer a parameter. The velocity may be made dimensionless with a characteristic speed U, the position vector with a characteristic particle "radius" a, and the temperature with some characteristic temperature. By these means the energy equation may be written in the nondimensional form

$$\underline{\nabla}^2\underline{T} - \text{Pé}^*\underline{\mathbf{v}}\cdot\underline{\nabla}\underline{T} = 0 \tag{299}$$

where

$$\text{Pé}^* = aU/\alpha \tag{300}$$

is the Péclet number based on "radius."

1. *Zero Péclet Number*

Assuming that the various dimensionless quantitites remain finite in the limit as $\text{Pé}^* \to 0$, Eq. (299) reduces to Laplace's equation. Thus, in dimensional form,

$$\nabla^2 T = 0 \tag{301}$$

This relation may be regarded as the counterpart of Stokes equations, Eqs. (7)–(8). Roughly speaking, Eq. (301) bears the same relationship to Eq. (299) at very small Péclet (and Reynolds) numbers as do Stokes equations to the complete Navier–Stokes equations at very small Reynolds numbers. Hence, many of the results of Section II pertaining to the solutions of Stokes equations have analogs in the theory of heat- (and mass-) transfer at asymptotically small Péclet numbers. It will suffice, therefore, to illustrate these analogs by a few salient examples.

Consider the case of heat transfer in the infinite space external to some body B. Suppose that the body is maintained at a constant uniform temperature T_B, and that the unperturbed temperature field at infinity is some arbitrarily prescribed solution of Laplace's equation, $T_\infty(\mathbf{r})$. Thus, one is to solve Eq. (301) so as to satisfy these boundary conditions. A conventional heat-transfer coefficient cannot be defined for this situation because of the spatially-variable nature of T_∞. However, by resorting to *symbolic* methods the rate of heat transfer per unit time from the body to the surrounding medium may be written in the symbolic form

$$Q = \hbar A(T_B - T_\infty) \tag{302}$$

in which A is the external surface area of the body and the "heat-transfer coefficient" $\hbar$ is a *scalar, symbolic, differential operator*. This relation is the analog of the comparable momentum-transfer relations (81)–(82). In particular, for a spherical particle of radius a embedded in a medium of thermal conductivity k, the symbolic Nusselt number (based on radius) is

$$\frac{\hbar a}{k} = \frac{\sinh(a\nabla_O)}{a\nabla_O} \tag{303}$$

in which O refers to the sphere center and $\nabla = (\nabla^2)^{1/2}$. This should be compared with Eqs. (87)–(88). By resorting to the infinite series expansion for the hyperbolic sine, and noting that $\nabla^2 T_\infty = 0$ everywhere, and that $A = 4\pi a^2$, one thus obtains

$$Q = 4\pi k a\{T_B - (T_\infty)_O\} \tag{304}$$

where the subscript O implies evaluation at the sphere center. This relation is the analog of Faxén's laws, Eqs. (89a)–(89b). As the series expansion (304) terminates after a finite number of terms, the sphere does not provide a convincing example of the significant advantages afforded by the use of a symbolic heat-transfer coefficient. Rather, the comparable expression for the ellipsoid [see also Eqs. (92a)–(92b)],

$$\mathscr{h}A = \frac{8\pi k}{\chi}\frac{\sinh D_O}{D_O} \tag{305}$$

illustrates the technique to better advantage. Here, χ and D are defined in Eqs. (62) and (93), respectively. The subscript O refers to evaluation at the center of the ellipsoid.

Equation (302) originates from the relation

$$Q = -k\int_B (T - T_\infty)\nabla\tau \cdot d\mathbf{S} \tag{306}$$

where the scalar field τ satisfies the following differential equation and boundary conditions:

$$\begin{aligned} &\nabla^2\tau = 0 \\ &\tau = 1 \quad \text{on } B \\ &\tau \to 0 \quad \text{at infinity} \end{aligned} \tag{307}$$

These relations are the precursors of Eqs. (303) and (305). Equation (306) is the counterpart of Eqs. (78)–(79), whereas the field τ defined by Eq. (307) is comparable to the intrinsic fields defined in Eqs. (17)–(20) and (25)–(28). The derivation of Eq. (306) depends upon the use of a reciprocal theorem, in the form of Green's second identity (L5a), in much the same way as the derivation of Eqs. (78)–(79) devolves upon the use of a comparable reciprocal theorem (B18). Fourier's law, $\mathbf{q} = -k\nabla T$, for the heat-flux vector plays a role comparable to Newton's law of viscosity, Eq. (3), in the former derivation.

The representation of the macroscopic properties of bodies by symbolic operators may be applied in other contexts. In electrostatics, for example, if we interpret Q as the charge on a conducting body, and T as the electric potential, then $\mathscr{h}A$ in Eq. (302) may be interpreted as the symbolic capacitance of the body, viewed as a condenser; k then plays the role of the dielectric constant (capacitivity) of the medium external to the body, this medium being assumed homogeneous and isotropic.

2. *Small, Nonzero Péclet Numbers*

The preceding development furnishes formulas for the rate of heat transfer in Stokes flow correctly to the zeroth order in the Péclet number. As Acrivos and Taylor (A1a) have so convincingly demonstrated in the case of streaming flows, higher-order terms in the Nusselt number-Péclet number expansion cannot be obtained by conventional perturbation schemes. Rather, such attempts encounter the same fundamental difficulty as is met in attempts to obtain higher-order terms, beyond Stokes law, in the drag coefficient-Reynolds number expansion, as discussed in Section III,A. In particular, no matter how small the Péclet number, the leading term in the expansion, that is to say, the solution of Eq. (301), does not furnish a uniformly valid solution of the energy equation (299). This leads to a Whitehead-type "paradox" analogous to that discussed in a hydrodynamic context in Section III,A. Again, the difficulty may be resolved by resorting to singular perturbation methods (Section III,C), utilizing separate, locally valid, inner and outer expansions of the temperature field, and asymptotically "matching" them in their common domain of validity, i.e., where $\text{Pé}^* \underline{r} = O(1)$—$\underline{r}$ being the dimensionless distance from the body.

By this technique Acrivos and Taylor obtained the following result[54] for streaming Stokes flow past a spherical particle of radius a maintained at uniform temperature in a fluid whose temperature is uniform at infinity:

$$\text{Nu} = 2 + \tfrac{1}{2}\,\text{Pé} + \tfrac{1}{4}\,\text{Pé}^2 \ln \text{Pé} + 0.03404\,\text{Pé}^2 + \tfrac{1}{16}\,\text{Pé}^3 \ln \text{Pé} + O(\text{Pé}^3) \tag{308}$$

in which

$$\text{Nu} = 2ha/k \tag{309}$$

is the average Nusselt number based on sphere diameter (h being the conventional heat-transfer coefficient), and

$$\text{Pé} = 2aU/\alpha \tag{310}$$

is the Péclet number based on sphere diameter (U being the speed of the stream at infinity). Each successive term, $f_n(\text{Pé})$, in Eq. (308) has the property that

$$\lim_{\text{Pé} \to 0} f_{n+1}(\text{Pé})/f_n(\text{Pé}) = 0$$

Equation (308) is the analog of Proudman and Pearson's (P11) result for the drag on a sphere at small, nonzero Reynolds numbers, quoted in Eq. (212).

[54] The leading term in the comparable asymptotic expansion for *large* Péclet numbers, i.e., $\text{Pé} \to \infty$, obtained by thermal boundary layer arguments (L10a, F11, A1a), is

$$\text{Nu} = 0.991(\text{Pé})^{1/3} + O(\text{Pé}^0)$$

Higher-order terms in this expansion are given explicitly by Acrivos and Goddard (A1).

The appearance of transcendental terms in these expansions is characteristic of the spatially nonuniform properties of the individual perturbation terms.

Equation (308) was generalized by Brenner (B18a) to particles of arbitrary shape (not necessarily *solid* particles), resulting in the following expansion:

$$\mathrm{Nu} = \mathrm{Nu}_0 + \tfrac{1}{8}\,\mathrm{Nu}_0{}^2\,\mathrm{P\acute{e}} + \tfrac{1}{16}\,\mathrm{Nu}_0{}^2 f_{0\alpha}\,\mathrm{P\acute{e}}^2 \ln \mathrm{P\acute{e}} + O(\mathrm{P\acute{e}}^2) \tag{311}$$

where

$$\mathrm{Nu} = \frac{hA}{2\pi k a} \tag{312a}$$

$$\mathrm{Nu}_0 = \frac{h_0 A}{2\pi k a} \tag{312b}$$

is a generalized Nusselt number based on the characteristic particle "diameter" $2a$ (A being the wetted area of the particle), and Pé is as defined in Eq. (310) with U the speed of the stream at infinity. The subscript "0" on the Nusselt number in Eq. (312b) refers to its value for a *stagnant fluid* i.e., for $\mathrm{P\acute{e}} = 0$. Furthermore,

$$f_{0\alpha} = |(F_0)_\alpha|/6\pi\mu a U \tag{313}$$

is the dimensionless Stokes drag on the particle, $(F_0)_\alpha$ being the dimensional drag on the body—that is, the component of the (Stokes) vector force on the particle in the direction of the stream velocity vector $\mathbf{U} = \boldsymbol{\alpha} U$. For a solid sphere of radius a ($\mathrm{Nu}_0 = 2$, $f_{0\alpha} = 1$), this correctly reproduces the first three terms of Eq. (308). The terms of $O(\mathrm{P\acute{e}}^2)$ and higher in Eq. (311) depend explicitly on the geometry of the particle and on the fluid-dynamical boundary conditions at its surface.

Equation (311) is the heat-transfer analog of the corresponding hydrodynamic formula, Eq. (232), expressing the drag on a particle of arbitrary shape in terms of the Stokes force. The utility of Eq. (311) lies in the fact that it permits one to predict the heat-transfer coefficient h for streaming flow past the body solely from a knowledge of the comparable coefficient h_0 for molecular conduction, and from a knowledge of the Stokes drag on the body. Inasmuch as the drag necessarily depends upon the orientation of the body relative to the free-stream velocity, Eq. (311) provides an explicit illustration of the fact that the heat-transfer coefficient varies with the orientation of the body.

The overall rate of heat transfer, from bodies of any shape past which fluid streams in Stokes flow, has the remarkable property of being invariant to reversal of the direction of the streaming flow at infinity (B26b); that is, for a particle of uniform surface temperature immersed in a fluid of different uniform temperature at infinity, the average Nusselt number is the same for flow past the particle in any specified direction, as for flow at the same speed in the opposite direction. What makes this result surprising is that the local temperature field, defined by Eq. (298), does not itself generally display such invariance. This theorem applies irrespectively of the magnitude of the Péclet number ($0 \leqslant \text{Pé} < \infty$) – even in the thermal boundary-layer regime, $\text{Pé} \to \infty$. As such, it contradicts intuition; for the detailed structure of the thermal boundary-layer for flow past nonspherical particles (lacking fore-aft symmetry relative to the direction of streaming) depends markedly upon the direction of flow. Equation (311), valid at small Pé, is concordant with this flow-reversal theorem. In fact, it was the prior existence of this result that originally suggested the general theorem. Inasmuch as proof of the theorem depends only upon the fact that, except for the algebraic sign, the local velocity field $\mathbf{v}$ is invariant to flow reversal, and requires only that the *normal* velocity component vanish at the particle surface, the general result also applies to other classes of " reversible " flows, e.g., streaming potential flow past the particle.

O'Brien (O1a) applied the Acrivos–Taylor analysis to the case where the Reynolds number, though small, is not identically zero as in the Stokes flow case. The analysis is vastly more complicated because, to any order in the Reynolds number, the velocity field $\mathbf{v}$ appearing in Eq. (298) is now expressed in terms of *two*, locally valid expansions [the inner and outer expansions of $\mathbf{v}$ given by the Proudman–Pearson analysis (P11)], rather than the single Stokes velocity field. For a solid spherical particle she obtains for small Pé and Re

$$\text{Nu} = 2 + \tfrac{1}{2}\,\text{Pé}(1 - \tfrac{17}{64}\,\text{Re}) + \cdots \tag{314}$$

where $\text{Re} = 2aU/\nu$ is the Reynolds number based on sphere diameter, ν being the kinematic viscosity. This result apparently applies to the case where

$$\text{Re} \ll \text{Pé} \ll 1 \tag{315a}$$

Alternatively, since $\text{Pé} = \text{Pr Re}$, where $\text{Pr} = \nu/\alpha$ is the Prandtl number, the range of applicability is

$$1 \ll \text{Pr} \ll (\text{Re})^{-1} \tag{315b}$$

An extension of the Acrivos–Taylor (A1a) heat transfer analysis to slip flow past the sphere is given by Taylor (T2a). Hartunian and Liu (H9a) and Taylor (T2b) include the effect of surface chemical reactions in their singular perturbation treatment of Stokes-flow mass transfer from spheres at small Péclet numbers.

B. Brownian Motion

In this subsection we present a general theory of the translational and rotational Brownian motions of rigid particles of arbitrary shape.

One of the important applications of Stokes law occurs in the theory of Brownian motion. According to Einstein (E1a) the translational and rotary diffusion coefficients for a spherical particle of radius a diffusing in a medium of viscosity μ are, respectively,

$$D_t = \frac{kT}{6\pi\mu a} \tag{316a}$$

$$D_r = \frac{kT}{8\pi\mu a^3} \tag{316b}$$

where k is Boltzmann's constant (the gas constant per molecule) and T is the absolute temperature. These relations presuppose that the diffusing solute particle is large compared with the mean-free path of the solvent molecules. Thus, from the point of view of the diffusing particle, the surrounding solvent medium may be regarded as continuous, and the conventional principles of continuum fluid mechanics applied. The minute dimensions of the solute particles insure that the pertinent translational and rotational Reynolds numbers will, on the average, be small, thereby bringing the phenomenon within the domain of a Stokes flow. The denominators $6\pi\mu a$ and $8\pi\mu a^3$ appearing above are the proportionality coefficients between the hydrodynamic force and translational velocity, and hydrodynamic torque and angular velocity, respectively.

In addition to Eqs. (316), Einstein also deduced the fundamental laws describing the mean-square linear and angular displacements of a free particle in a small time interval Δt, these being

$$\overline{(\Delta x)^2} = 2D_t \Delta t \tag{317a}$$

$$\overline{(\Delta \phi)^2} = 2D_r \Delta t \tag{317b}$$

Because of the symmetry of the sphere, no coupling occurs between the translational and rotational Brownian movements, leading to the further relation

$$\overline{(\Delta x)(\Delta \phi)} = 0 \tag{317c}$$

This subsection is concerned with the generalization of Eqs. (316) and (317) to particles of arbitrary shape, and with certain fundamental analogies relating the general theory of Brownian motion to the corresponding general theory of hydrodynamic resistance at small Reynolds numbers. For the most part, what follows is a brief summary and commentary on results reported by Brenner (B25a, B26a).

In the presence of a nonuniformity in concentration a diffusive flux occurs, resulting in the creation of entropy. The rate of irreversible entropy production is, in general, a homogeneous quadratic function of the gradients of concentration in both physical and orientation space, the proportionality coefficient being an appropriately generalized diffusion tensor. From another point of view, the (mean) ensuing motion (both translational and rotational) of the macroscopic solute particles relative to the solvent results in the dissipation of mechanical energy. The rate of such dissipation is a homogeneous quadratic function of the velocities, the coefficient of proportionality being, apart from the viscosity, the hydrodynamic resistance tensor. Because of the intimate connection between entropy production and dissipation rate,[55] and of the interrelation between diffusion currents and particle velocities, there exists an almost perfect analogy between the macroscopic theory of diffusion in fluids and of the hydrodynamic resistance of particles. The analogy is deeper than is implied by quantitative relationships of the Stokes–Einstein type, Eq. (316). Rather, the analogy extends to the underlying structure of the fundamental equations governing both phenomena. The remarkable similarities are best brought out by considering the diffusion of particles of arbitrary shape. For it is here, in the most general case, that the analogies are most striking.

Consider a large number of identical rigid particles of arbitrary shape distributed throughout a fluid. Specification of the instantaneous "configuration" of each particle requires knowledge of an appropriate set of six independent parameters $(q^1, q^2, \ldots, q^6)$. These constitute the generalized curvilinear coordinates of the particle in the six-dimensional "configuration space." Three of these, say (q^1, q^2, q^3), are required to specify the position of the particle in "physical space" (e.g., the Cartesian coordinates of some arbitrary origin O fixed in the particle, relative to an origin fixed in the fluid). The remaining three (q^4, q^5, q^6) are required to specify the orientation of the particle in "orientation space" (e.g., the Eulerian angles of a set of Cartesian axes fixed in the particle, relative to a set of Cartesian axes fixed in the fluid).

The geometrical properties of the configuration space are entirely determined by its metric tensor. If ds is the distance between two adjacent points in the space, then

$$ds^2 = g_{ij}\, dq^i\, dq^j \tag{318}$$

where the $g_{ij} = g_{ji}$ are the covariant components of the metric tensor, the latter being a positive-definite form. Unless the contrary is explicitly stated, we utilize the Einstein summation convention on a repeated index when it

[55] The relation is, in fact (L5b, pp. 382–387),

$$\dot{S} = \frac{1}{T}\dot{\Phi}$$

where $\dot{S}$ is the time rate of irreversible entropy production, $\dot{\Phi}$ the rate of mechanical energy dissipation, and T the absolute temperature of the system.

appears as both a covariant and contravariant index. Latin indices range over the integers from 1 to 6. At a given instant of time t the number of particles whose coordinates lie within the volume element between q^1 and $q^1 + dq^1$, q^2 and $q^2 + dq^2, \ldots, q^6$ and $q^6 + dq^6$ is denoted by $\sigma(q^1, q^2, \ldots, q^6; t)\, dV$, where the volume element is given by $dV = \sqrt{g}\, dq^1\, dq^2 \cdots dq^6$ in which $g = \det \|g_{ij}\|$; σ is thus the local density of particles in the configuration space; that is, the number of particles per unit volume of physical space per unit volume of orientation space. Integration over all orientations yields the ordinary number density ρ—the number of solute particles per unit volume of physical space.

The physical laws ultimately derived are necessarily independent of the particular q-parametrization selected. No conceptual difficulties arise in an invariant treatment of the "translation group" in physical space, and conventional vector methods suffice. However, an invariant treatment of the "rotation group" (G2a) in orientation space demands a somewhat more sophisticated knowledge of group-theoretic methods. Accordingly, it is convenient to formulate the general theory in terms of a particular parametrization which allows application of ordinary Cartesian vector methods to the latter as well. Choose an arbitrary origin O fixed in the body and select any set of right-handed Cartesian axes, Ox^1, Ox^2, Ox^3, fixed in the particle, with their origin at O. As a result of Brownian motion and/or convective transport the particle will, in some small time interval dt, be displaced from its original position and orientation. The infinitesimal change in the position of point O in physical space may be represented by the vector

$$d\mathbf{r} = \mathbf{i}_1\, dx^1 + \mathbf{i}_2\, dx^2 + \mathbf{i}_3\, dx^3 \tag{319}$$

where dx^1, dx^2, dx^3 denote the projections of the displacement vector onto the body axes, and $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors parallel to these axes. Similarly, the change of orientation may be represented by the infinitesimal rotation vector $d\boldsymbol{\phi}$ defined in the following manner: The magnitude of this vector is the angle $|d\phi|$ through which the body must be rotated from its original orientation to produce its final orientation. The direction of this vector is along the instantaneous axis of rotation in a right-handed sense (L7a, p. 18). If $d\phi^1$, $d\phi^2$, $d\phi^3$ denote the projections of this vector onto the body axes, then

$$d\boldsymbol{\phi} = \mathbf{i}_1\, d\phi^1 + \mathbf{i}_2\, d\phi^2 + \mathbf{i}_3\, d\phi^3 \tag{320}$$

The quantity $d\boldsymbol{\phi}$ is a vector only for infinitesimal rotations, but not for finite rotations (G9e).

In terms of this representation we have

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (d\phi^1)^2 + (d\phi^2)^2 + (d\phi^3)^2 \tag{321}$$

so that for the body coordinates, $g_{ij} = \delta_{ij}$. It is convenient to set

$$\|dq^1, dq^2, dq^3\| = \|dx^1, dx^2, dx^3\| \equiv d\mathbf{r} \tag{322}$$

$$\|dq^4, dq^5, dq^6\| = \|d\phi^1, d\phi^2, d\phi^3\| \equiv d\boldsymbol{\phi} \tag{323}$$

As finite rotations cannot be represented by this vector method, there exist no quantities ϕ^1, ϕ^2, ϕ^3 for which $d\phi^1, d\phi^2, d\phi^3$ are the differentials; that is to say, the latter are inexact differentials. Hence, the curvilinear coordinates q^4, q^5, q^6 are defined only in a *local* sense, and any operations performed with them should be regarded as purely symbolic (L7a, p. 109; L5b, pp. 138, 149).

The translational flux vector, $\overset{(t)}{\mathbf{J}}_O$, is defined in the usual way in terms of the number of points O crossing a unit area of physical space per unit time, irrespective of the orientations of the individual particles. Analogously, the rotary flux vector, $\overset{(r)}{\mathbf{J}}$, is defined in terms of the number of particles crossing a unit area of orientation space per unit time, irrespective of the positions of the particles in physical space. The former flux depends upon the choice of origin O, whereas the latter does not—hence the difference in subscript notation. Each of the two fluxes consists of two parts: (i) a convective contribution arising either from macroscopic motion of the fluid (translational or rotational) or from the action of external forces or torques on the individual particles, and (ii) a diffusional contribution arising from the thermal agitation of the particles. Thus

$$\overset{(t)}{\mathbf{J}}_O = {}_c\overset{(t)}{\mathbf{J}}_O + {}_d\overset{(t)}{\mathbf{J}}_O \tag{324a}$$

$$\overset{(r)}{\mathbf{J}} = {}_c\overset{(r)}{\mathbf{J}} + {}_d\overset{(r)}{\mathbf{J}} \tag{324b}$$

The convective fluxes are[56]

$${}_c\overset{(t)}{\mathbf{J}}_O = \sigma \mathbf{U}_O \tag{325a}$$

$${}_c\overset{(r)}{\mathbf{J}} = \sigma \boldsymbol{\omega} \tag{325b}$$

[56] If the fluid is macroscopically at rest, convective fluxes can arise only from the action of external forces and torques on the particles. The particle velocities can then be obtained from the given forces and torques by solving Eqs. (38) and (39) for $\mathbf{U}_O$ and $\boldsymbol{\omega}$, bearing in mind that the hydrodynamic and external forces acting on a particle are in equilibrium. Equivalently, these velocities may be obtained by inverting the matrix equation (49),

$$\|\mathscr{U}_O\| = -\frac{1}{\mu}\|\mathscr{K}_O\|^{-1}\|\mathscr{F}_O\|$$

If the fluid is not macroscopically at rest, but is, rather, undergoing some net motion, say a shearing flow, then $\mathbf{U}_O$ and $\boldsymbol{\omega}$ may be obtained from Eqs. (109)–(110). If no external forces and torques act in the latter case the particle velocities are then given by Eqs. (120) and (121).

in which

$$\mathbf{U}_O = \mathbf{i}_1\dot{x}^1 + \mathbf{i}_2\dot{x}^2 + \mathbf{i}_3\dot{x}^3 \tag{326a}$$

and

$$\boldsymbol{\omega} = \mathbf{i}_1\dot{\phi}^1 + \mathbf{i}_2\dot{\phi}^2 + \mathbf{i}_3\dot{\phi}^3 \tag{326b}$$

are, respectively, the translational velocity of point O and the angular velocity of the particle. The overdot denotes a derivative with respect to time.

The translational and rotational diffusion fluxes can be shown to be linear functions of the gradients, the general relation being of the form (B26a):

$$ {}_d\overset{(t)}{\mathbf{J}}_O = -\overset{(t)}{\mathbf{D}}_O \cdot \frac{\partial \sigma}{\partial \mathbf{r}} - \overset{(c)}{\mathbf{D}}_O{}^{\dagger} \cdot \frac{\partial \sigma}{\partial \boldsymbol{\phi}_O} \tag{327}$$

$$ {}_d\overset{(r)}{\mathbf{J}} = -\overset{(c)}{\mathbf{D}}_O \cdot \frac{\partial \sigma}{\partial \mathbf{r}} - \overset{(r)}{\mathbf{D}} \cdot \frac{\partial \sigma}{\partial \boldsymbol{\phi}_O} \tag{328}$$

where $\overset{(t)}{\mathbf{D}}_O$ is the translational diffusivity dyadic at O, $\overset{(r)}{\mathbf{D}}$ is the rotational diffusivity dyadic, and $\overset{(c)}{\mathbf{D}}_O$ is the coupling diffusivity dyadic at O.[57] As usual, the subscript O indicates that the quantity to which it is affixed depends upon the choice of origin, O. The dyadic diffusivities (second-rank tensor diffusion coefficients) appearing in Eqs. (327) and (328) are constant relative to body axes. These diffusion dyadics are intrinsic entities, being independent of the particular choice of body axes. The "direct" dyadics, $\overset{(t)}{\mathbf{D}}_O$ and $\overset{(r)}{\mathbf{D}}$, are symmetric, positive-definite, true tensors, whereas the "cross" dyadic, $\overset{(c)}{\mathbf{D}}_O$, is a pseudotensor. The latter is associated with any screw-like geometric properties of the diffusing particle, and vanishes for bodies lacking such properties, e.g., ellipsoidal particles, providing that the origin is properly chosen. The coupling diffusivity dyadic is a measure of the degree of interaction between the translational and rotational Brownian motions.

Equations (327)–(328) are the complete analogs of the hydrodynamic

[57] Here,

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{i}_1 \frac{\partial}{\partial x^1} + \mathbf{i}_2 \frac{\partial}{\partial x^2} + \mathbf{i}_3 \frac{\partial}{\partial x^3}$$

$$\frac{\partial}{\partial \boldsymbol{\phi}} = \mathbf{i}_1 \frac{\partial}{\partial \phi^1} + \mathbf{i}_2 \frac{\partial}{\partial \phi^2} + \mathbf{i}_3 \frac{\partial}{\partial \phi^3}$$

the former derivative being equivalent to the ordinary ∇-operator.

equations (38)–(39). Accordingly, as in the case of Eq. (49), one may resort to the matrix forms

$$\|{}_c\mathscr{J}_O\| = \sigma\|\mathscr{U}_O\| \tag{329}$$

in which

$$\|\mathscr{U}_O\| = \begin{Vmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \\ \dot{\phi}^1 \\ \dot{\phi}^2 \\ \dot{\phi}^3 \end{Vmatrix} \tag{330}$$

and

$$\|{}_d\mathscr{J}_O\| = -\|\mathscr{D}_O\|\;\|\mathrm{grad}_O\,\sigma\| \tag{331}$$

where

$$\|\mathscr{J}_O\| = \begin{Vmatrix} \overset{(t)}{J}{}^1(O) \\ \overset{(t)}{J}{}^2(O) \\ \overset{(t)}{J}{}^3(O) \\ \overset{(r)}{J}{}^1 \\ \overset{(r)}{J}{}^2 \\ \overset{(r)}{J}{}^3 \end{Vmatrix}, \qquad \|\mathrm{grad}_O\,\sigma\| = \begin{Vmatrix} \partial\sigma/\partial x^1 \\ \partial\sigma/\partial x^2 \\ \partial\sigma/\partial x^3 \\ \partial\sigma/\partial {\phi_O}^1 \\ \partial\sigma/\partial {\phi_O}^2 \\ \partial\sigma/\partial {\phi_O}^3 \end{Vmatrix} \tag{332a,b}$$

and $\|\mathscr{D}_O\|$ is the 6×6 partitioned diffusivity matrix

$$\|\mathscr{D}_O\| = \begin{Vmatrix} \left\| \overset{(t)}{\mathbf{D}}_O \right\| & \left\| \overset{(c)}{\mathbf{D}}_O \right\|^{\dagger} \\ \left\| \overset{(c)}{\mathbf{D}}_O \right\| & \left\| \overset{(r)}{\mathbf{D}} \right\| \end{Vmatrix} \tag{333}$$

in which

$$\left\| \overset{(t)}{\mathbf{D}}_O \right\| = \begin{Vmatrix} \overset{(t)}{D}{}^{11}(O) & \overset{(t)}{D}{}^{12}(O) & \overset{(t)}{D}{}^{13}(O) \\ \overset{(t)}{D}{}^{21}(O) & \overset{(t)}{D}{}^{22}(O) & \overset{(t)}{D}{}^{23}(O) \\ \overset{(t)}{D}{}^{31}(O) & \overset{(t)}{D}{}^{32}(O) & \overset{(t)}{D}{}^{33}(O) \end{Vmatrix}, \quad \text{etc.} \tag{334}$$

In certain applications it is more convenient to express these relations in the general tensor form

$$J^i = {}_cJ^i + {}_dJ^i \tag{335}$$

in which the J^i ($i = 1, 2, \ldots, 6$) are the contravariant components of the flux vector in configuration space. For simplicity we have suppressed the O-indices. In this form the convective flux is

$$ {}_cJ^i = \sigma \dot{q}^i \tag{336}$$

whereas the diffusion flux is

$$ {}_dJ^i = -D^{ij} \frac{\partial \sigma}{\partial q^j} \tag{337}$$

in which the D^{ij} are the contravariant components of the diffusivity tensor. These tensor relations hold in any system of coordinates as, indeed, do the corresponding dyadic and matrix relations when properly interpreted. It should be clearly understood, however, that the various diffusion dyadics, matrices, and tensors are constant only when referred to body axes.

Equations (327)–(328) or, equivalently, their matrix and tensor counterparts, constitute the generalization of Fick's law of diffusion to the six-dimensional configuration space. The derivation of this generalization (B25a, B26a) proceeds along lines originally set forth by Einstein (E1a). In the absence of sources or sinks, conservation of probability density in configuration space leads to the following continuity relation (B25a):

$$\frac{\partial \sigma}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} (\sqrt{g} J^i) = 0 \tag{338}$$

the second term being the expression for the divergence (of the flux vector) in general tensor form. Equations (335)–(338) lead to a partial differential equation for the density σ. The resulting equation, expressed in a convenient system of space axes, rather than body axes, is given by Brenner (B26a). Space axes are the more useful of the two systems in the actual solution of initial- and boundary-value problems.

The degree of completeness of the analogies between Eqs. (331) and (49) is quite remarkable; $\|\mathscr{D}_O\|$ and $\|\mathscr{K}_O\|$ are each symmetric, positive-definite forms, as are their direct submatrices too. The positive-definiteness of $\|\mathscr{D}_O\|$ stems from the positivity of the rate of irreversible entropy production. In contrast to the proof of the symmetry of the hydrodynamic resistance matrix (B22),[58] the corresponding proof of the symmetry of the diffusion matrix is trivial. The latter may be taken to be symmetric by *definition* since its antisymmetric part gives rise to no observable macroscopic physical consequence.

[58] In their volume on "Mechanics," Landau and Lifshitz (L7a, p. 76) state, in effect, that symmetry of the resistance matrix cannot be demonstrated by purely macroscopic mechanical arguments. Their statement is, however, refuted by the macroscopic symmetry proof offered by Brenner (B22).

As implied by the subscript notation in Eqs. (327)–(328), the translational and coupling diffusivity dyadics vary with choice of origin. This dependence can be quantitatively established. By invoking appropriate kinematic arguments, it can be shown (B26a) that, whereas $\overset{(r)}{\mathbf{J}}$ and $\partial\sigma/\partial\mathbf{r}$ are independent of choice of origin,

$$\overset{(t)}{\mathbf{J}}_P = \overset{(t)}{\mathbf{J}}_O - \mathbf{r}_{OP} \times \overset{(r)}{\mathbf{J}}$$

$$\frac{\partial\sigma}{\partial\boldsymbol{\phi}_P} = \frac{\partial\sigma}{\partial\boldsymbol{\phi}_O} + \frac{\partial\sigma}{\partial\mathbf{r}} \times \mathbf{r}_{OP}$$

in which O and P are any arbitrary points fixed in the particle, and $\mathbf{r}_{OP}$ is the vector drawn from O to P. These relations are the analogs of those given in the sentence following Eq. (52). By these means one arrives at the following expressions for the origin-dependence of the various diffusivity dyadics:

$$\overset{(c)}{\mathbf{D}}_P = \overset{(c)}{\mathbf{D}}_O + \overset{(r)}{\mathbf{D}} \times \mathbf{r}_{OP} \tag{339}$$

$$\overset{(t)}{\mathbf{D}}_P = \overset{(t)}{\mathbf{D}}_O - \mathbf{r}_{OP} \times \overset{(r)}{\mathbf{D}} \times \mathbf{r}_{OP} + \overset{(c)}{\mathbf{D}}_O{}^{\dagger} \times \mathbf{r}_{OP} - \mathbf{r}_{OP} \times \overset{(c)}{\mathbf{D}}_O \tag{340}$$

The rotational diffusivity dyadic is origin-independent. These origin-displacement theorems are the counterparts of the comparable hydrodynamic relations set forth in Eqs. (51) and (52).

As in the hydrodynamic case, there exists a unique origin, say R, at which the coupling diffusivity dyadic is symmetric; that is,

$$\overset{(c)}{\mathbf{D}}_R = \overset{(c)}{\mathbf{D}}_R{}^{\dagger} \tag{341}$$

If the value of the coupling diffusivity dyadic is known at some point O, and if the value of the rotational diffusivity dyadic is also known, the location of R (relative to O) may be determined from the relation

$$\mathbf{r}_{OR} = -\left[\left(\mathbf{I} : \overset{(r)}{\mathbf{D}}\right)\mathbf{I} - \overset{(r)}{\mathbf{D}}\right]^{-1} \cdot \boldsymbol{\varepsilon} : \overset{(c)}{\mathbf{D}}_O \tag{342}$$

which is the analog of Eq. (54). In general, if Eq. (350) applies, the hydrodynamic center of reaction is identical to the diffusive center of reaction (B26a). In any event, the two points can be shown to be the same for particles possessing various types of geometric symmetries, even if Eq. (350) is not applicable.

The symmetric dyadic $\overset{(c)}{\mathbf{D}}_R$ is identically zero for a large class of symmetrical particles, including spheres and ellipsoids. For such bodies there is no coupling between the translational and rotary Brownian motions, and Eqs. (327) and (328) adopt the simple, "uncoupled" forms

$$ {}_d\overset{(t)}{\mathbf{J}}_R = -\overset{(t)}{\mathbf{D}}_R \cdot \frac{\partial\sigma}{\partial\mathbf{r}} \tag{343a} $$

$$ {}_d\overset{(r)}{\mathbf{J}} = -\overset{(r)}{\mathbf{D}} \cdot \frac{\partial\sigma}{\partial\boldsymbol{\phi}_R} \tag{343b} $$

analogous to Eqs. (55a, b), *et seq.* With regard to various types of geometric symmetry that particles may possess, it is obvious that the symmetry properties of the diffusion dyadics $\overset{(t)}{\mathbf{D}}_R$ and $\overset{(r)}{\mathbf{D}}$ are analogous to those of the resistance dyadics $\overset{(t)}{\mathbf{K}}$ and $\overset{(r)}{\mathbf{K}}_R$, respectively; whereas $\overset{(c)}{\mathbf{D}}_R$ and $\overset{(c)}{\mathbf{K}}_R$ are comparable as regards their symmetry properties. Accordingly, one is led *inter alia* to the conclusion that in regard to their diffusive properties, there exist helicoidally isotropic particles (B25a):

$$ \overset{(t)}{\mathbf{D}}_R = \mathbf{I}D_t \tag{344a} $$

$$ \overset{(r)}{\mathbf{D}}_R = \mathbf{I}D_r \tag{344b} $$

$$ \overset{(c)}{\mathbf{D}}_R = \mathbf{I}D_c \tag{344c} $$

as well as spherically isotropic particles, characterized by setting $D_c = 0$ in the above. The scalar diffusivities D_t and D_r, and pseudoscalar diffusivity D_c satisfy the inequalities

$$ D_t > 0 \tag{345a} $$

$$ D_r > 0 \tag{345b} $$

$$ D_t D_r > D_c^2 \tag{345c} $$

D_c being positive for right-handed bodies, and conversely. These relations are comparable to those given in Eq. (74). The existence of a coupling diffusivity term, relating to the screw-like geometric properties of particles, seems to have escaped notice before now.

Correlations between the linear displacement (of the origin O) and angular displacement in a given time interval due to Brownian motion may be expressed in terms of a generalized form of Eq. (317). It is found that (B26a)

$$ \begin{aligned} \overline{\Delta\mathbf{r}_O\Delta\mathbf{r}_O} &= 2\overset{(t)}{\mathbf{D}}_O\Delta t, \qquad & \overline{\Delta\mathbf{r}_O\Delta\boldsymbol{\phi}} &= 2\overset{(c)}{\mathbf{D}}_O{}^{\dagger}\Delta t \\ \overline{\Delta\boldsymbol{\phi}\Delta\mathbf{r}_O} &= 2\overset{(c)}{\mathbf{D}}_O\Delta t, \qquad & \overline{\Delta\boldsymbol{\phi}\Delta\boldsymbol{\phi}} &= 2\overset{(r)}{\mathbf{D}}\Delta t \end{aligned} \tag{346} $$

provided that $|\Delta\boldsymbol{\phi}| \ll \pi$. The average displacements themselves are identically zero:

$$\overline{\Delta \mathbf{r}_O} = \mathbf{0}, \qquad \overline{\Delta\boldsymbol{\phi}} = \mathbf{0}$$

Equivalently, Eq. (346) may also be expressed as the single matrix equation

$$\overline{\|\Delta \mathcal{Q}_O\|^{\dagger} \, \|\Delta \mathcal{Q}_O\|} = 2\| \mathcal{D}_O\| \Delta t \tag{347}$$

where $\|\Delta \mathcal{Q}_O\|$ is the row matrix

$$\|\Delta \mathcal{Q}_O\| = \|\Delta x^1(O), \Delta x^2(O), \Delta x^3(O), \Delta\phi^1, \Delta\phi^2, \Delta\phi^3\| \tag{348}$$

and its transpose, $\|\Delta \mathcal{Q}_O\|^{\dagger}$, is the corresponding column matrix.

The preceding relations apply to rigid particles of any shape. If the origin is chosen at the center of reaction, cross-correlations between the linear and angular displacements will vanish whenever the symmetry of the particle is such that $\overset{(c)}{\mathbf{D}}_R = \mathbf{0}$. This occurs, for example, in the case of ellipsoidal particles. In the interesting case where the particles are helicoidally isotropic, the correlation relations take the highly symmetric form

$$\overline{\Delta x^{\alpha} \Delta x^{\beta}} = 2 D_t \Delta t \tag{349a}$$

$$\overline{\Delta \phi^{\alpha} \Delta \phi^{\beta}} = 2 D_r \Delta t \tag{349b}$$

$$\overline{\Delta x^{\alpha} \Delta \phi^{\beta}} = 2 D_c \Delta t \tag{349c}$$

where α, $\beta = 1, 2, 3$. When the particle is spherically isotropic $D_c = 0$, leading to relations of the type given in Eq. (317), originally obtained by Einstein for spherical particles.

The general theory of Brownian motion set forth thus far is a self-contained, purely phenomenological theory which does not depend upon any special hydrodynamic assumptions. If, however, following Einstein, it is further assumed that, on the average, the diffusing particle experiences a hydrodynamic force governed by Stokes equations, the diffusion matrix is then found to be (B26a)

$$\|\mathcal{D}_O\| = \frac{kT}{\mu} \|\mathcal{K}_O\|^{-1} \tag{350}$$ [59]

[59] The general form of this relationship agrees with the well-known Nernst–Planck–Einstein formula, according to which the diffusion coefficient is given by the relation

Diffusion Coefficient $= kT$ (Mobility)

where the mobility is the velocity imparted to the particle by a unit external force; for, as is clear from Eq. (49), the generalized mobility is $\mu^{-1}\|\mathcal{K}_O\|^{-1}$.

This constitutes the generalization of the Stokes–Einstein formulas (316). It applies for any choice of origin. The positive-definite character of the resistance matrix assures the existence of its inverse, as required by the preceding relation.

Equation (350) furnishes a wholly independent proof of the symmetry and positive-definiteness of the diffusivity matrix, for these characteristics now follow from the comparable properties of the hydrodynamic resistance matrix.

Expressed in terms of the individual resistance and diffusion dyadics, Eq. (350) may be written as

$$\overset{(t)}{\mathbf{D}}_O = \frac{kT}{\mu}\left(\overset{(t)}{\mathbf{K}} - \overset{(c)}{\mathbf{K}}_O{}^\dagger \cdot \overset{(r)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(c)}{\mathbf{K}}_O\right)^{-1} \tag{351a}$$

$$\overset{(r)}{\mathbf{D}} = \frac{kT}{\mu}\left(\overset{(r)}{\mathbf{K}}_O - \overset{(c)}{\mathbf{K}}_O \cdot \overset{(t)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(c)}{\mathbf{K}}_O{}^\dagger\right)^{-1} \tag{351b}$$

$$\overset{(c)}{\mathbf{D}}_O = -\frac{kT}{\mu}\overset{(r)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(c)}{\mathbf{K}}_O \cdot \left(\overset{(t)}{\mathbf{K}} - \overset{(c)}{\mathbf{K}}_O{}^\dagger \cdot \overset{(r)}{\mathbf{K}}{}_O^{-1} \cdot \overset{(c)}{\mathbf{K}}_O\right)^{-1} \tag{351c}$$

In the case of a particle for which no coupling occurs at the center of reaction, these reduce to

$$\overset{(t)}{\mathbf{D}}_R = \frac{kT}{\mu}\overset{(t)}{\mathbf{K}}{}^{-1} \tag{352a}$$

$$\overset{(r)}{\mathbf{D}} = \frac{kT}{\mu}\overset{(r)}{\mathbf{K}}{}_R^{-1} \tag{352b}$$

$$\overset{(c)}{\mathbf{D}}_R = \mathbf{0} \tag{352c}$$

For example, in the case of an ellipsoidal particle, one easily finds from Eqs. (58)–(59) that

$$\overset{(t)}{\mathbf{D}}_R = \frac{kT}{16\pi\mu}\left[\mathbf{e}_1\mathbf{e}_1(\chi + a_1{}^2\alpha_1) + \mathbf{e}_2\mathbf{e}_2(\chi + a_2{}^2\alpha_2) + \mathbf{e}_3\mathbf{e}_3(\chi + a_3{}^2\alpha_3)\right] \tag{353a}$$

and

$$\overset{(r)}{\mathbf{D}} = \frac{3kT}{16\pi\mu}\left(\mathbf{e}_1\mathbf{e}_1\,\frac{a_2{}^2\alpha_2 + a_3{}^2\alpha_3}{a_2{}^2 + a_3{}^2} + \mathbf{e}_2\mathbf{e}_2\,\frac{a_3{}^2\alpha_3 + a_1{}^2\alpha_1}{a_3{}^2 + a_1{}^2} + \mathbf{e}_3\mathbf{e}_3\,\frac{a_1{}^2\alpha_1 + a_2{}^2\alpha_2}{a_1{}^2 + a_2{}^2}\right) \tag{353b}$$

In the case of helicoidally isotropic particles it follows from Eqs. (73), (344), and (351) that

$$D_t = \frac{kT}{\mu} \frac{K_r}{K_t K_r - K_c^2} \tag{354a}$$

$$D_r = \frac{kT}{\mu} \frac{K_t}{K_t K_r - K_c^2} \tag{354b}$$

$$D_c = -\frac{kT}{\mu} \frac{K_c}{K_t K_r - K_c^2} \tag{354c}$$

In accordance with Eq. (74) the above denominators are essentially positive. It is also clear that the inequalities set forth in Eq. (345) are satisfied.

When the particles are spherically isotropic these reduce to the forms $D_c = 0$ and

$$D_t = \frac{kT}{\mu K_t} \tag{355a}$$

$$D_r = \frac{kT}{\mu K_r} \tag{355b}$$

of which the original Stokes–Einstein formulas (316) are special cases.

If all particle orientations are assumed equally probable, one finds upon integrating over orientation space that the usual form of Fick's law in physical space applies:

$$\frac{\partial \rho}{\partial t} = \bar{D}_t \nabla^2 \rho \tag{356}$$

where

$$\rho = \iiint \sigma \, d\phi^1 \, d\phi^2 \, d\phi^3 \tag{357}$$

is the number density of particles in physical space. For nonskew particles the mean diffusivity is the scalar (P4a)

$$\bar{D}_t = \frac{1}{3} \operatorname{trace} \left\| \overset{(t)}{\mathbf{D}}_R \right\|$$

whence, from Eq. (352a),

$$\bar{D}_t = \frac{1}{3} \frac{kT}{\mu} \operatorname{trace} \left\| \overset{(t)}{\mathbf{K}} \right\|^{-1} \tag{358}$$

This should be compared with the comparable formula for the average resistance of a nonskew particle near the end of Section II,C,1. For helicoidally isotropic particles the mean diffusivity is identical to D_t given in Eq. (354a) (B25a).

C. Summary and Commentary

An attempt has been made in this article to critically survey the field of low Reynolds number flows, with particular regard to the hydrodynamic resistance of particles in this regime. A remarkable burgeoning of interest in such problems has occurred within the past decade. Significant advances have been recorded on both the theoretical and experimental sides, with the former gains far outdistancing the latter in scope. Problems which would have been impossible to solve rigorously before the advent of singular perturbation techniques are now being regularly solved, though hardly in a routine fashion; insight, intuition, inspiration, and ingenuity are still the order of the day.

For those interested in direct engineering applications of the material covered by this review, the perspective from which many of the more general results set forth here should be viewed is, perhaps, best illustrated by an example: The resistance of any solid particle to translational and rotational motions in Stokes flow may be completely calculated from knowledge of a set of 21 scalar coefficients (Section II,C,1). While it seems highly improbable to expect that all these coefficients could be experimentally measured in practice, except perhaps in the trivial case of highly symmetrical bodies for which many of the coefficients vanish identically, this does not detract from the conceptual advantages of knowing exactly how much one does not know. Having an ideal goal against which the extent of present knowledge can be gaged permits a rational decision as to how to optimize one's investment of time, effort, and money in the pursuit of additional data. Furthermore, with the development of high-speed digital computers it may soon be possible to *calculate* all these coefficients for any given body (O1b). The general theory provides a rigorous framework into which such knowledge may be embedded.

Use of symbolic "drag coefficients" (Section II,C,2) and symbolic heat- and mass-transfer "coefficients" (Section IV,A) furnishes a unique method for describing the intrinsic, interphase transport properties of particles for a wide variety of boundary conditions. Here, the particle resistance is characterized by a partial differential operator that represents its intrinsic resistance to vector or scalar transfer, independently of the physical properties of the fluid, the state of motion of the particle, or of the unperturbed velocity or temperature fields at infinity. Though restricted as yet in applicability, the general ideas underlying the existence of these operators appear capable of extension in a variety of ways.

A recurrent theme arising throughout the analysis pertains to the screw-like properties of particles and of their intrinsic right- and left-handedness (Sections II,C,1; II,C,2; III,C and IV,B). Such properties reflect an inseparable coupling between the translational and rotational motions of the particle.

Helicoidally isotropic particles furnish the simplest examples of bodies manifesting screw-like behavior. These particles are isotropic, in that their properties are the same in all directions. Yet they possess a sense, and spin as they settle in a fluid. These ideas are likely to be of interest to microbiologists, biophysicists, geneticists, and others in the life sciences for whom handedness and life are intimately intertwined. The microscopic dimensions of the objects of interest to them insures *ipso facto* that the motion takes place at very small Reynolds numbers. Readers interested in an elementary but broad survey of sense in the physical and biological sciences are referred to Gardner's delightful book "The Ambidextrous Universe" (G1).

First-order corrections to the Stokes force on a particle, arising from wall- or inertial-effects, can be directly expressed in terms of the Stokes force on the body in the absence of such effects. Thus, with regard to wall-effects in the Stokes regime, Eq. (135) expresses the force experienced by a particle falling in, say, a circular cylinder, in terms of the comparable force experienced by the particle when falling with the same velocity and orientation in the unbounded fluid. Equation (139) expresses a similar relationship for the torque on a rotating particle in a circular cylinder, as does Eq. (166) for the first-order interaction between two particles in an unbounded fluid in terms of the properties of the individual particles. Analogously, Eq. (234) expresses the inertial correction to the Stokes drag force in terms of the Stokes force itself. A comparable relationship exists (Section IV,A) between the heat-transfer coefficient at small, nonzero Péclet numbers and the heat-transfer coefficient at zero Péclet number—that is, the coefficient for conduction heat transfer. Finally, Eqs. (78)–(79) (or their symbolic operator counterparts) permit direct calculation of the Stokes force and torque experienced by a particle in an arbitrary field of flow solely from knowledge of the elementary solutions of Stokes equations for translation and rotation of the particle in a fluid at rest at infinity. The utility of already available knowledge is thus greatly extended by the existence of such relations. It permits one whose interests lie entirely in the macroscopic manifestation of the motion, e.g., the force and torque on the body, to bypass the oftentimes difficult problem of obtaining a detailed solution of the equations of motion, and to proceed directly to the computation of the force and torque on the body from the prescribed boundary conditions alone. The calculation is thereby reduced to a quadrature.

The contents of this review may be read simultaneously from two different points of view. First and foremost it may be regarded as a compendium of recent advances in low Reynolds number flows. Secondly, from a pedagogic viewpoint it may be profitably used to illustrate the direct application of invariant techniques, that is, vector-polyadic and tensor methods, to a class of physical problems. Because of the relative simplicity and rich variety of physical problems associated with low Reynolds number motions, intuitive arguments may be employed to gain insight into the nature of polyadics and tensors; the role played by the concept of direction as a primitive entity is brought out here to a degree not usually found in standard works on tensor analysis.

Acknowledgments

The author is grateful to Dr. S. G. Mason and the Pulp and Paper Research Institute of Canada for their encouragement and continued interest in this work. Part of the writing of this chapter was supported by a grant from the National Science Foundation (Grant No. NSF GK-56), to whom the author wishes to express his thanks. A debt of gratitude is also due Dr. Raymond G. Cox for his many stimulating and penetrating comments. During the all-too-brief tenure of his postdoctoral fellowship in the Chemical Engineering Department at New York University, he was a constant source of inspiration and fresh insight.

Nomenclature[60,61]

Roman letters

- a Representative particle length (4); radius of sphere (29); polar radius of prolate spheroid (75)–(76)
- a_1, a_2, a_3 Lengths of semiaxes of an ellipsoid (57)
- A Cross sectional area (143); external surface area (302)
- A, B, C Constants (90b)–(91)
- $\mathbf{A}$ Constant dyadic (97)
- b Equatorial radius of prolate spheroid (75)–(76); distance from cylinder axis to sphere center (135)–(136)
- B Refers to surface of body (11)
- (c) Over symbol denoting coupling (38)
- c One-half the distance between focii of a prolate spheroid (75)–(76)
- C Dimensionless Stokes force (242)
- $\mathbf{C}_k$ Constant k-adic (98)–(99)
- d Particle diameter (198)–(199)
- ds Distance element in configuration space (318)
- $d\mathbf{S}$ Directed element of surface area pointing into fluid (40)
- dV Volume element in configuration space (318)–(319)
- $d\phi^i$ Cartesian component of vector $d\boldsymbol{\phi}$ (320)
- $d\boldsymbol{\phi}$ Differential angular displacement vector (320)
- $d\Omega$ Element of surface area on a unit sphere (91b)–(92)
- D Symbolic operator $= (D^2)^{1/2}$ (92a)
- D^2 Ellipsoidal differential operator (93)

[60] Numbers in parentheses refer to the equation number in which the symbol first appears or is first defined. When the first appearance of the symbol occurs within the text, rather than in a displayed equation, this is so indicated by giving the equation numbers lying on either side of that portion of the text. For example, (75)–(76) indicates that the first appearance of the symbol occurs somewhere between Eqs. (75) and (76).

[61] An underlined symbol is nondimensional. It has the same meaning as its dimensional, nonunderlined counterpart.

D_c, D_r, D_t	Coupling, rotation, and translational diffusion coefficients for an isotropic body (316), (344)
D^{ij}	Contravariant component of diffusion tensor (337)
$\overset{(c)}{\mathbf{D}}$, $\overset{(r)}{\mathbf{D}}$, $\overset{(t)}{\mathbf{D}}$	Coupling, rotational, and translational diffusivity dyadics (327), (328)
$\mathbf{e}$	Normalized eigenvector of $\mathbf{K}$ (43)–(44); unit vector (166)
$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$	Normalized eigenvectors of $\mathbf{K}$ (44); unit vectors parallel to principal axes of ellipsoid (58)
$\mathbf{e}_1$	Unit vector along symmetry axis of body of revolution (72)
$\mathbf{e}_{12}$	Unit vector from center of particle 1 to center of particle 2 (165)
E_{ijk}	Cartesian component of $\mathbf{E}$ (227)
$\mathbf{E}$	Intrinsic rate of strain triadic (221)
$f(x)$	Analytic function of x (91b)–(92)
$f(\beta)$	Wall-correction factor for nonaxial particle (136)
$f_k(\theta, \phi)$	Surface spherical harmonic of degree k (64)
$f_n(\text{Pé})$	Function of Péclet number (310)–(311)
$f_n(R)$	Function of Reynolds number (212)–(213)
$f_{0\alpha}$	Dimensionless Stokes drag in the α direction (313)
$\mathbf{f}$	External force per unit volume (180)
$\mathbf{f}(\mathbf{r})$	Analytic function of position (90a)
(F)	Over symbol denoting force (100)
F	Component of hydrodynamic force (130)
F_D	Drag force in axial direction (245)–(246)
F_L	Lift force in radial direction (245)–(246)
$(F_0)_\alpha$	Stokes drag in α direction (232)
F_∞'	Infinite medium force based on approach velocity (141)
$F_{\text{I}}(\beta)$, $F_{\text{II}}(\beta)$, $F_{\text{III}}(\beta)$	Functions of β (288), (289), (290)
$\mathbf{F}$	Vector hydrodynamic force exerted by fluid on particle (38)
$\mathbf{F}_D$	Drag force vector (215)
$\mathbf{F}_L$	Lift force vector (216)
$\mathbf{F}_w$	Force exerted by fluid on duct walls (144)
g	Magnitude of local acceleration of gravity vector $\mathbf{g}$ (295); determinant of fundamental tensor g_{ij} (318)–(319)
$g(\zeta)$	Function of ζ (75)–(76)
g_{ij}	Covariant component of metric tensor (318)
$\Vert\text{grad}\Vert$	Matrix of generalized vector gradient $\partial/\partial q^i$ in configuration space (332b)
$\mathbf{g}$	Local acceleration of gravity vector (145)
h	Distance from particle center to plane (131); center-to-center distance (165); heat-transfer coefficient (309)
H	Gap width (132)–(133)
$\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$	Right-handed triad of mutually perpendicular unit vectors (footnote 3)
$I_{n+1/2}$	Modified Bessel function of the first kind (98)–(99)
$\mathbf{I}$	Dyadic idemfactor (3)
J^i	Tensor component of flux vector $\mathbf{J}$ (335)
$\overset{(r)}{\mathbf{J}}$, $\overset{(t)}{\mathbf{J}}$	Rotational and translational flux vectors (323)–(324)
${}_c\mathbf{J}$, ${}_d\mathbf{J}$	Convective and diffusive flux vectors (325), (327)
k	Permeability of isotropic porous medium (173); thermal conductivity (303); Boltzmann constant (316)
$\mathbf{k}$	Dyadic permeability coefficient (171)
K	Eigenvalue of $\mathbf{K}$ (43)–(44)
K_1, K_2, K_3	Eigenvalues of $\mathbf{K}$ (44)

K_c, K_r, K_t Coupling, rotation, and translational resistances for an isotropic body (71)

$K^{(r)} K^{(t)}, K^{(c)}$, Scalar and pseudoscalar resistance coefficients for a porous medium (footnote 19)

$K_{\parallel}, K_{\perp}$ Respective components of $\mathbf{K}$ parallel and perpendicular to symmetry axis (72)

K_{ij} Cartesian tensor component of $\mathbf{K}$ (45)–(46)

$K_{ij}{}^{(kl)}$ (k, l) Cartesian tensor component of the dyadic $\mathbf{K}_{ij}$ (159)

$\mathbf{K}$ Generic symbol for a resistance dyadic (43)–(44)

$\overset{(c)}{\mathbf{K}}, \overset{(r)}{\mathbf{K}}, \overset{(t)}{\mathbf{K}}$ Coupling, rotation, and translation dyadics (38), (39)

$\mathbf{K}^{(t)}, \mathbf{K}^{(c)}, \mathbf{K}^{(r)}$ Dyadic, triadic and tetradic, resistance coefficients, respectively, for a porous medium (footnote 19)

$\mathbf{K}_t, \mathbf{K}_c''$ Dyadic resistance coefficients for a porous medium (174)–(175)

$\mathbf{K}_c', \mathbf{K}_r$ Triadic resistance coefficients for a porous medium (174)–(175)

$\mathbf{K}_\infty$ Resistance dyadic in an unbounded fluid (134)

$\mathbf{K}_1, \mathbf{K}_2$ Resistance dyadics for particles 1 and 2, respectively, in an unbounded fluid (166)

$\mathbf{K}_{ij}$ Resistance dyadic in a multiparticle system (148), (149)

l Characteristic distance of particle from boundary (122)–(123); radius of circular cylinder (126); characteristic apparatus dimension (179)–(180)

L Length of duct (145)–(146)

$\mathbf{M}$ Vector moment of forces (173)–(174)

$\boxed{n}$ Polydot multiplication symbol (98)–(99)

N_{Po} Dimensionless power number (198)–(199)

N_{Re} Reynolds number (198)–(199)

Nu Average Nusselt number (309)

O Arbitrary origin affixed to particle (9)

(O) Refers to evaluation at origin O (45)

p Pressure (1)

p^0 Unperturbed pressure field (89)–(90)

p^+ Additional pressure field (145)–(146)

$p^\cdot$ Static pressure field (145)–(146)

$P_k(\cos\theta)$ Legendre polynomial of degree k and argument $\cos\theta$ (235)–(236)

$P_n, P_n{}^m$ Legendre functions (68)–(69)

Pé Péclet number based on "diameter" (310)

Pé* Péclet number based on "radius" (300)

Pr Prandtl number (315a)–(315b)

$\mathbf{P}$ Vector "pressure" field (17), (25), or (285a)

ΔP^+ Additional pressure drop vector (142)–(143)

q^i Generalized coordinate in configuration space (317)–(318)

$\dot{q}^i$ Generalized contravariant velocity component in configuration space (336)

$\dfrac{\partial}{\partial q^i}$ Generalized vector gradient in configuration space (337)

$\mathbf{q}$ Superficial velocity vector in a porous medium (171); heat-flux vector (307)–(308)

Q Rate of heat transfer (302)

$\mathbf{Q}$ Constant dyadic (94)

(r) Over symbol denoting rotation (13)

r Spherical coordinate (29); distance $|\mathbf{r}|$ from some origin (29)

$\mathbf{r}$ Position vector of a point (4)

$\mathbf{r}^n$ Polyadic (99)

$\mathbf{r}_{OP}$ Vector drawn from O to P (51)

$\dfrac{\partial}{\partial \mathbf{r}}$ Symbolic representation for vector gradient in physical space (footnote 57)

$\Delta\mathbf{r}$ Linear displacement vector in physical space (346)

R Particle Reynolds number (5); cylindrical coordinate (245)

R^* Generalized particle Reynolds number (241)

R_o Cylinder radius (245)

Re Reynolds number based on sphere diameter (314)

Re_p $2a|\mathscr{U}_p|/\nu$; particle Reynolds number based on axial slip velocity (252)

Re_s $(2a)^2 S/\nu$; shear Reynolds number (269)–(270)

Re_t $2R_o V_m/\nu$; tube Reynolds number (251)

Re_ω $(2a)^2|\omega|/\nu$; rotational Reynolds number (269)–(270)

Re_∞ $2a|U_\infty|/\nu$; Reynolds number based on free-fall velocity in an unbounded fluid at rest (266)

$\mathbf{R}$ Relative position vector of two points (194)

S Wetted surface area of particle (70); velocity gradient (269a)

$\dot{S}$ Time rate of irreversible entropy production (footnote 55)

$\bar{S}$ Intensity of shear (122)

S_1 Unit sphere (91b)–(92)

S_{ij} Cartesian tensor component of dyadic $\mathbf{S}$ (97)–(98)

$\mathbf{S}$ Rate of shear dyadic (108)

(t) Over symbol denoting translation (11)

t Time (1)

Δt Small interval of time (317)

$\mathbf{t}$ Torque per unit volume (173)–(174)

(T) Over symbol denoting torque (101)

T Component of vector torque $\mathbf{T}$ (75); temperature (298); absolute temperature (316)

T_B Temperature at surface of body (302)

T_i Component of vector torque $\mathbf{T}$ parallel to principal axis of ellipsoid (97)–(98)

T_∞ Undisturbed temperature field at infinity (302)

$\mathbf{T}$ Vector hydrodynamic torque exerted by fluid on particle (39)

u Component of vector $\mathbf{u}$ (143)

$\mathbf{u}$ Unperturbed vector flow field (77)

U Particle speed; component of $\mathbf{U}$ (135); axial component of sphere velocity in direction of net flow (245)–(246)

U_L Radial (lift) velocity component (245)–(246)

U_∞ Terminal settling speed in an unbounded fluid (138); component of settling velocity vector $\mathbf{U}_\infty$ parallel to direction of net flow (247)–(248)

$\mathbf{U}$ Particle velocity vector (9); stream velocity vector (199)

$\mathbf{U}_\infty$ Settling velocity vector in an unbounded fluid at rest (247)–(248)

v Particle volume (145)–(146)

$\mathbf{v}$ Local fluid velocity vector (1)

$\mathbf{v}'$ Local fluid velocity vector measured relative to an observer translating with particle (269a)

V Representative particle speed (4); volume (318)–(319)

V_m Mean velocity of flow (143)

V_{ij} Cartesian tensor component of dyadic $\mathbf{V}$ (227)

$\mathbf{V}$ Dyadic "velocity" field, (17), (25), or (285a)

$\mathbf{V}_m$ Mean velocity of flow vector (141)–(142)

W An eigenvalue of $\mathbf{W}$ (135)

$\mathbf{W}$ Wall-effect dyadic (134)

x_1, x_2, x_3 Cartesian coordinates measured parallel to principal axes (57)

x_1', x_2', x_3' Cartesian coordinates measured relative to sphere center (268)–(269)

$\underline{x}_1, \underline{x}_2, \underline{x}_3$ Cartesian coordinates made dimensionless with cylinder radius (287)–(288)

$\underline{\tilde{x}}_1, \underline{\tilde{x}}_2, \underline{\tilde{x}}_3$ Nondimensional Cartesian coordinates measured relative to a variable origin (287)–(288)

x^1, x^2, x^3 Cartesian coordinates (318)–(319)

$\dot{x}^i$ Cartesian velocity component (326)

Δx^i Cartesian component of vector $\Delta \mathbf{r}$ (317), (349)

x, y, z Cartesian coordinates (195)–(196)

Script roman letters

$\|\mathscr{D}\|$ 6×6 Diffusion matrix (333)

$\|\mathscr{F}\|$ Wrench matrix (46), (160a)

$\mathscr{h}$ Symbolic heat transfer coefficient (302)

$\|\mathscr{J}\|$ Flux matrix in configuration space (332a)

$\|_c\mathscr{J}\|, \|_d\mathscr{J}\|$ Convective and diffusive flux matrices in configuration space (329), (331)

$\|\mathscr{K}\|$ Resistance matrix (48); grand resistance matrix (161)

$\mathscr{K}_{ijk}$ Cartesian tensor component of $\boldsymbol{\mathscr{K}}$ (227)

$\boldsymbol{\mathscr{K}}$ Inertial resistance triadic (226)

$\mathscr{p}$ Mean pressure in Darcy's law (171)

$\mathscr{p}^*$ "Proper" pressure in Darcy's law (174)

$\|\Delta\mathscr{Q}\|$ 6-dimensional matrix of generalized displacements Δq^i (348)

$\mathscr{U}_p$ Axial component of particle slip velocity (246)

$\|\mathscr{U}\|$ Screw-velocity matrix (47), (160b)

$\mathscr{V}$ Characteristic speed (217)

Greek letters

α Thermal diffusivity (298)

$\alpha_1, \alpha_2, \alpha_3$ Functions of a_1, a_2, a_3 (61)

$\boldsymbol{\alpha}$ Unit vector parallel to $\mathbf{U}$ (231)–(232)

β b/l or b/R_o; fractional distance from cylinder axis to particle center (136)

β^* Stable equilibrium value of β (244)–(245)

γ Interfacial tension (122a)

Γ Volume of space external to particle (227); entire volume of space within circular cylinder (288)

δ Dirac delta function (193)–(194); small parameter (68)–(69)

δ_{jk} Kronecker delta (45)–(46)

$\Delta(\lambda)$ Function of λ (63)

ϵ Small dimensionless deformation parameter (64)

ε_{jkl} Permutation symbol (footnote 3)

$\boldsymbol{\varepsilon}$ Isotropic triadic (footnote 3)

ζ Prolate spheroidal coordinate (75)–(76)

η Viscosity ratio (122)

θ Polar angle in spherical coordinates (64)

λ Dummy variable of integration (61)

Λ Dimensionless ratio (294)

μ Viscosity (1)

ν Angle defined in connection with helicoidal symmetry (72)–(73); kinematic viscosity (244)–(245)

ξ^* Generalized ratio of representative particle-to-boundary dimensions (243)

$\boldsymbol{\pi}$ Stress dyadic (3)

$\boldsymbol{\Pi}$ Intrinsic "stress" triadic (34), (37)

ρ Fluid density (1)

ρ_p Particle density (244)–(245)

ρ_m Mean density of a suspension (145)

$\Delta\rho$ Fluid-particle density difference (295)

σ Wall-correction factor (130); density of particles in configuration space (318)–(319)

- τ Characteristic time (4); fundamental temperature field (307)
- ϕ Azimuthal angle in spherical coordinates (64); fractional volume of solid particles in a suspension (145)–(146)
- $\dot{\phi}^i$ Cartesian component of angular velocity (326b)
- $\Delta\phi^i$ Cartesian component of vector $\Delta\boldsymbol{\phi}$ (317), (349)
- $\Delta\boldsymbol{\phi}$ Angular displacement vector in orientation space (346)
- $\dfrac{\partial}{\partial\boldsymbol{\phi}}$ Symbolic representation for vector gradient in orientation space (footnote 57)
- Φ Rate of mechanical energy dissipation (footnote 55)
- χ Function of a_1, a_2, a_3 (62)
- ψ Stream function (75)
- ω Angular speed; component of $\boldsymbol{\omega}$ (75)
- ω_i^f Component of angular velocity vector $\boldsymbol{\omega}_f$ parallel to principal axis of ellipsoid (97)–(98)
- ϖ Cylindrical coordinate (75)
- $\boldsymbol{\omega}$ Angular velocity vector (9)
- $\boldsymbol{\omega}_f$ Angular velocity of fluid particle (107)

GERMAN LETTERS

- $\mathfrak{F}$ Symbolic dyadic force operator (83)
- ${}_{n+2}\mathfrak{K}$ Constant resistance polyadic (102), (103)
- $\mathfrak{L}_n^{(0)}$, $\mathfrak{L}_n^{(2)}$ Symbolic scalar and dyadic operators, respectively (98)
- $\mathfrak{S}$ Cartesian component of $\boldsymbol{\mathfrak{S}}$ (119)
- $\boldsymbol{\mathfrak{S}}$ Constant shear resistance triadic (111), (112)
- $\mathfrak{T}$ Symbolic dyadic torque operator (84)

OTHER SYMBOLS

- ∇ Symbolic scalar operator $= (\nabla^2)^{1/2}$ (87)
- ∇^2 Laplace operator (1)
- $\boldsymbol{\nabla}$ Vector nabla operator (1)
- $\boldsymbol{\nabla}^n$ Polyadic differential operator (98)–(99)
- $\Box$ Vector differential operator (92b)
- $\|\ \|$ Matrix brackets (45)–(46)
- — Under symbol denoting dimensionless quantity (4); over symbol denoting mean value (75)–(76), (317)
- $\cdot$ Over symbol denoting derivative with respect to time (326)
- $\cdot$ Dot product (1)
- $:$ Double-dot product (54)
- $\times$ Cross product (9)
- $\sim$ Over symbol denoting "stretched" variable (204)

SUBSCRIPTS

- B Body
- c Coupling
- c Presubscript denoting convective
- d Presubscript denoting diffusive
- D Drag
- ext External
- f Fluid
- int Internal
- i, j, k, l, m, n Summation or tensor indices
- L Lift force
- m Mean value
- $n+2$ Presubscript denoting degree of a polyadic
- O Arbitrary origin
- p Particle
- P Any point
- r Rotation
- R Center of reaction

s Shear
t Translation, or tube
w Wall
α Component of vector in $\boldsymbol{\alpha}$ direction
ω Angular spin
$\parallel$ Parallel
$\perp$ Perpendicular
∞ Infinite medium
0 Stokes field
-2 Presubscript denoting a term homogeneous in $\underline{r}^{-2}$

SUPERSCRIPTS

f Fluid
$i, j, k, l, \ldots$ Tensor indices
$\alpha, \beta, \ldots$ Tensor indices in 3-space
0 Unperturbed flow
-1 Inverse (reciprocal) dyadic
(1), (2), (3) Cartesian tensor components of a vector
(11), (12), ..., (33) Cartesian tensor components of a dyadic
1, 2 Presuperscript denoting vectors which are reversed and invariant, respectively, to a reversal of direction of motion
† Transposition operator
∞ Unbounded fluid
$+$ Additional value, above and beyond value in the unperturbed flow
• Static fluid

REFERENCES

A1. Acrivos, A., and Goddard, J. D., *J. Fluid Mech.* **23,** 273 (1965).
A1a. Acrivos, A., and Taylor, T. D., *Phys. Fluids* **5,** 387 (1962).
A2. Acrivos, A., and Taylor, T. D., *Chem. Eng. Sci.* **19,** 445 (1964).
A3. Andersson, O., *Svensk Papperstid.* **59,** 540 (1956); **60,** 153 (1957).
A4. Andersson, O., *Svensk Papperstid.* **60,** 341 (1957).
A5. Aris, R., "Vectors, Tensors and the Basic Equations of Fluid Mechanics," p. 30. Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
B1. Bammi, J. R., Mutual influence of two freely falling spherical particles and the effects of a plane vertical boundary on a single spherical particle. M.S. Dissertation, Univ. Iowa, Iowa City, 1950.
B2. Becker, H. A., *Can. J. Chem. Eng.* **37,** 85 (1959).
B3. Belinfante, D. C., *Proc. Cambridge Phil. Soc.* **58,** 405 (1962).
B4. Berker, R., *in* "Encyclopedia of Physics: Fluid Dynamics II" (S. Flügge and C. Truesdell, eds.), Vol. 8, Part 2, p. 200. Springer, Berlin, 1963.
B4a. Bhatnagar, P. L., and Rajeswari, G. K., *Compt. Rend.* **256,** 3823 (1963).
B4b. Bhatnagar, R. K., *Proc. Indian Acad. Sci.* **A60,** 99 (1964).
B5. Bickley, W. G., *Phil. Mag.* [7] **25,** 746 (1938).
B6. Block, H. D., "Introduction to Tensor Analysis." Merrill, Columbus, Ohio, 1962.
B7. Block, R. B., *J. Appl. Phys.* **11,** 635 (1940).
B8. Bohlin, T., *Trans. Roy. Inst. Technol., Stockholm* **155,** 1 (1960); see comments by Faxén, H., *Kolloid-Z.* **167,** 146 (1959).
B9. Bratukhin, Iu. K., *J. Appl. Math. Mech.* (English Transl. of *Prikl. Mat. Mekh.*), **25,** 1286 (1961)
B10. Breach, D. R., *J. Fluid Mech.* **10,** 306 (1961).
B11. Brenner, H., A theoretical study of slow viscous flow through assemblages of spherical particles. Eng. Sc.D. Dissertation, New York University, New York, 1957.
B12. Brenner, H., *Phys. Fluids* **1,** 338 (1958).

B13. Brenner, H., *J. Fluid Mech.* **6**, 542 (1959).
B14. Brenner, H., *Chem. Eng. Sci.* **16**, 242 (1961).
B15. Brenner, H., *J. Fluid Mech.* **11**, 604 (1961).
B16. Brenner, H., *J. Fluid Mech.*, **12**, 35 (1962).
B17. Brenner, H., *Chem. Eng. Sci.* **17**, 435 (1962).
B18. Brenner, H., *Chem. Eng. Sci.* **18**, 1 (1963).
B18a. Brenner, H., *Chem. Eng. Sci.* **18**, 109 (1963).
B19. Brenner, H., *Chem. Eng. Sci.* **18**, 557 (1963).
B20. Brenner, H., *J. Fluid Mech.* **18**, 144 (1964).
B21. Brenner, H., *Chem. Eng. Sci.* **19**, 519 (1964).
B22. Brenner, H., *Chem. Eng. Sci.* **19**, 599 (1964).
B23. Brenner, H., *Chem. Eng. Sci.* **19**, 631 (1964).
B24. Brenner, H., *Chem. Eng. Sci.* **19**, 703 (1964).
B25. Brenner, H., *Appl. Sci. Res.* **A13**, 81 (1964).
B25a. Brenner, H., *J. Colloid Sci.* **20**, 104 (1965).
B26. Brenner, H., *Chem. Eng. Sci.* **21**, 97 (1966).
B26a. Brenner, H., Coupling between the translational and rotational Brownian motions of rigid particles of arbitrary shape: II. General theory. *J. Coll. Interface Sci.* (in press).
B26b. Brenner, H., On the invariance of the heat-transfer coefficient to flow reversal in Stokes flow past arbitrary particles (to be published).
B27. Brenner, H., and Cox, R. G., *J. Fluid Mech.* **17**, 561 (1963).
B28. Brenner, H., and Happel, J., *J. Fluid Mech.* **4**, 195 (1958).
B29. Brenner, H., and Sonshine, R. M., *Quart. J. Mech. Appl. Math.* **17**, 55 (1964).
B29a. Bretherton, F. P., *J. Fluid Mech.* **12**, 591 (1962).
B30. Bretherton, F. P., *J. Fluid Mech.* **14**, 284 (1962).
B31. Bretherton, F. P., *J. Fluid Mech.* **20**, 401 (1964).
B32. Brinkman, H. C., *Verhandel. Koninkl. Ned. Akad. Wetenschap.*, **50**, 618, 860 (1947).
B33. Brinkman, H. C., *Physica* **13**, 447 (1947).
B34. Brinkman, H. C., *Appl. Sci. Res.* **A1**, 27 (1947); **A1**, 81 (1948).
B35. Brinkman, H. C., *Research* (London) **2**, 190 (1949).
B36. Broersma, S., *J. Chem. Phys.* **32**, 1632 (1960).
B37. Burgers, J. M., *Verhandel. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Sect. I* (Chapter 3) **16**, 113 (1938).
B38. Burgers, J. M., *Verhandel. Koninkl. Ned. Akad. Wetenschap.* **44**, 1045, 1177 (1941); **45**, 9, 126 (1942).
C1. Caswell, B., and Schwarz, W. H., *J. Fluid Mech.* **13**, 417 (1962).
C2. Chaffey, C. E., Brenner, H., and Mason, S. G., *Rheol. Acta* **4**, 56 (1965).
C3. Chaffey, C. E., Brenner, H., and Mason, S. G., *Rheol. Acta* **4**, 64 (1965).
C4. Chang, I-Dee, *J. Appl. Math. Phys.* (*ZAMP*). **12**, 56 (1961).
C5. Chang, I-Dee, *J. Appl. Math. Phys.* (*ZAMP*). **14**, 134 (1963).
C6. Chapman, S., and Cowling, T. G., "The Mathematical Theory of Non-Uniform Gases." Cambridge Univ. Press, London and New York, 1961.
C7. Chester, W., *J. Fluid Mech.* **13**, 557 (1962).
C7a. Childress, S., *J. Fluid Mech.* **20**, 305 (1964).
C8. Chow, P. S.-H., A generalized drag coefficient-Reynolds number plot for non-spherical particles. M.S. Dissertation, Univ. Minnesota, Minneapolis, Minnesota, 1964.
C9. Chow, P. S.-H., and Brenner, H., A generalized drag coefficient-Reynolds number plot for non-spherical particles (to be published).

C10. Chowdhury, K. C. R., and Fritz, W., *Chem. Eng. Sci.* **11**, 92 (1959).
C11. Christopherson, D. G., and Dowson, D., *Proc. Roy. Soc.* **A251**, 550 (1959).
C12. Collins, R. E., "Flow of Fluids through Porous Media." Reinhold, New York, 1961.
C13. Collins, R. E., *in* "Modern Chemical Engineering" (A. Acrivos, ed.), Vol. 1, p. 307. Reinhold, New York, 1963.
C14. Collins, W. D., *Mathematika* **2**, 42 (1955).
C15. Collins, W. D., *Quart. J. Mech. Appl. Math.* **12**, 232 (1959).
C16. Collins, W. D., *Mathematika* **10**, 72 (1963).
C17. Cox, R. G., *J. Fluid Mech.* **23**, 273 (1965).
C18. Cox, R. G., and Brenner, H., The lateral migration of solid particles in Poiseuille flows (to be published).
C19. Cox, R. G., and Brenner, H., The motion of a sphere near a plane wall at small Reynolds numbers (to be published).
C20. Cox, R. G., and Brenner, H., Effect of finite boundaries on the Stokes resistance of an arbitrary particle: Part III. Translation and rotation (to be published).
C21. Cunningham, E., *Proc. Roy. Soc.* **A83**, 357 (1910).
D1. Dahler, J. S., and Scriven, L. E., *Nature* **192**, 36 (1961).
D2. Dahler, J. S., and Scriven, L. E., *Proc. Roy. Soc.* **A275**, 504 (1963).
D3. Datta, S. K., *Appl. Sci. Res.* **A11**, 47 (1962).
D3a. Day, J. T., and Genetti, W. E., Motion of spinning particles in shear fields. B.S. thesis, Univ. Utah, Salt Lake City, Utah, 1964.
D4. Dean, W. R., and O'Neill, M. E., *Mathematika* **10**, 13 (1963).
D4a. deGroot, S. R., and Mazur, P., "Non-Equilibrium Thermodynamics." North-Holland Publ., Amsterdam, 1962.
D4b. Denson, C. D., Particle migration in shear fields. Ph.D. thesis, Univ. Utah, Salt Lake City, Utah, 1965; see also Denson, C. D., Christiansen, E. B., and Salt, D. L., *A.I.Ch.E. (Am. Inst. Chem. Engrs.) J.* **12**, 589 (1966).
D4c. Debye, P., and Bueche, A. M., *J. Chem. Phys.* **16**, 573 (1948).
D5. DiFrancia, G. T., *Boll. Unione Mat. Ital.* [3] **5**, 273 (1950).
D6. Drew, T. B., "Handbook of Vector and Polyadic Analysis." Reinhold, New York, 1961.
D7. Duffing, G., *Z. angew. Math. Mech. (ZAMM)* **13**, 366 (1933).
E1. Eichhorn, R., and Small, S., *J. Fluid Mech.* **20**, 513 (1964).
E1a. Einstein, A., "Investigations on the Theory of the Brownian Movement," (R. Fürth, ed.). Dover, New York, 1956.
E2. Ericksen, J. L., *Trans. Soc. Rheol.* **4**, 29 (1960).
E3. Eveson, G. F., *Brit. J. Appl. Phys.* **11**, 88 (1960).
F1. Famularo, J., Theoretical study of sedimentation of dilute suspensions in creeping motion. Eng. Sc.D. Dissertation, New York University, New York, 1962; see also Famularo, J., and Happel, J., *A.I.Ch.E. (Am. Inst. Chem. Engrs,) J.* **11**, 981 (1965).
F2. Faxén, H., *Arkiv Mat. Astron. Fysik* **18**, 1 (1924).
F3. Fayon, A. M., Effect of a cylindrical boundary on rigid spheres suspended in a moving viscous liquid. Eng. Sc.D. Dissertation, New York University, New York, 1959.
F4. Fayon, A. M., and Happel, J., *A.I.Ch.E. (Am. Inst. Chem. Engrs.) J.* **6**, 55 (1960).
F5. Fidleris, V., Determination of wall-effect for spheres falling axially in cylindrical vessels. Ph.D. Dissertation, Univ. Nottingham, Nottingham, England, 1958.
F6. Fidleris, V., and Whitmore, R. L., *Brit. J. Appl. Phys.* **12**, 490 (1961).

F7. Forgacs, O. L., and Mason, S. G., *J. Colloid Sci.* **14**, 457 (1959).
F8. Forgacs, O. L., Robertson, A. A., and Mason, S. G. "Fundamentals of Papermaking Fibres," p. 447. British Paper and Board Makers' Assoc., Kenley, Surrey, England, 1958.
F9. Foster, R. D., and Slattery, J. D., *Appl. Sci. Res.* **A12**, 213 (1963).
F10. Frater, K. R., *J. Fluid Mech.* **20**, 369 (1964).
F11. Friedlander, S. K., *A.I.Ch.E.* (*Am. Inst. Chem. Engrs.*) *J.* **3**, 43 (1957).
F12. Friedman, B., "Principles and Techniques of Applied Mathematics." Wiley, New York, 1956.
F13. Fuchs, N. A. "The Mechanics of Aerosols." Pergamon Press, New York, 1964.
G1. Gardner, M., "The Ambidextrous Universe: Left, Right, and the Fall of Parity." Basic Books, New York, 1964.
G1a. Garstang, T. E., *Proc. Roy. Soc.* **A142**, 491 (1933).
G2. Gasparian, A. M., and Zaminian, A. A., *Aikakan SSR Gritertionneri Akad. Zeikaitsner* **26**, 39 (1958).
G2a. Gel'fand, I. M., Minlos, R. A., and Shapiro, Z. Ya., "Representations of the Rotation and Lorentz Groups and Their Applications." Macmillan, New York, 1963.
G3. Gibbs, J. W., and Wilson, E. B., "Vector Analysis." Dover, New York, 1960.
G4. Giesekus, H., *Rheol. Acta* **2**, 101 (1962).
G5. Giesekus, H., *Rheol. Acta* **3**, 59 (1963).
G5a. Giesekus, H., *Proc. Intern. Symp. Second-order Effects Elasticity, Plasticity, Fluid Dynamics, Haifa, 1962*, pp. 533–584. MacMillan (Pergamon) Press, New York, 1964.
G5b. Goldman, A. J., Cox, R. G., and Brenner, H., Slow viscous motion of two identical arbitrarily oriented spheres through a viscous fluid. *Chem. Eng. Sci.* (in press); see also Goldman, A. J., Investigations in low Reynolds number fluid-particle dynamics. Ph.D. Dissertation, New York University, New York, 1966.
G5c. Goldman, A. J., Cox, R. G., and Brenner, H., The Stokes resistance of an arbitrary particle—Part VI: Terminal motion of a settling particle (to be published); see also Goldman, A. J., Investigations in low Reynolds number fluid-particle dynamics. Ph.D. Dissertation, New York University, New York, 1966.
G5d. Goldman, A. J., Cox, R. G., and Brenner, H., Slow viscous motion of a sphere parallel to a plane wall—I. Motion through a quiescent fluid. *J. Fluid Mech.* (in press); see also Goldman, A. J., Investigations in low Reynolds number fluid-particle dynamics. Ph.D. Dissertation, New York University, New York, 1966.
G5e. Goldman, A. J., Cox, R. G., and Brenner, H., Slow viscous motion of a sphere parallel to a plane wall—II. Motion in a Couette flow. *J. Fluid Mech.* (in press); see also Goldman, A. J., Investigations in low Reynolds number fluid-particle dynamics. Ph.D. Dissertation, New York University, New York, 1966.
G6. Goldshtik, M. A., *Inzh.-Fiz. Zh. Akad. Nauk Belorussk SSR* **3**, 79 (1960).
G7. Goldsmith, H. L., and Mason, S. G., *Nature* **190**, 1095 (1961).
G8. Goldsmith, H. L., and Mason, S. G., *J. Fluid Mech.* **12**, 88 (1962).
G9. Goldsmith, H. L., and Mason, S. G., *J. Colloid Sci.* **17**, 448 (1962).
G9a. Goldsmith, H. L., and Mason, S. G., *Bibliotheca Anat.* **4**, 462 (1964).
G9b. Goldsmith, H. L., and Mason, S. G., *Bibliotheca Anat.* (in press).
G9c. Goldsmith, H. L., and Mason, S. G., *Biorheology* **3**, 33 (1965).
G9d. Goldsmith, H. L., and Mason, S. G., *in* "Rheology: Theory and Applications" (F. R. Eirich, ed.), Vol. IV. Academic Press, New York, 1966.

G9e. Goldstein, H., "Classical Mechanics," pp. 124–132. Addison-Wesley, Reading, Massachusetts, 1950.

G10. Goldstein, S., *Proc. Roy. Soc.* **A123**, 225 (1929).

G11. Goldstein, S., ed., "Modern Developments in Fluid Dynamics," Vol. 1, p. 83. Oxford Univ. Press, London and New York, 1938.

G12. Gupta, S. C., *J. Appl. Math. Phys.* (*ZAMP*) **8**, 257 (1957).

H1. Haberman, W. L., Flow about a sphere rotating in a viscous liquid inside a coaxially rotating cylinder. David W. Taylor Model Basin Rept. No. 1578. U.S. Navy Dept., Washington, D.C., 1961.

H2. Haberman, W. L., *Phys. Fluids* **5**, 625, 1136 (1962).

H3. Haberman, W. L., and Sayre, R. M., Motion of rigid and fluid spheres in stationary and moving liquids inside cylindrical tubes. David W. Taylor Model Basin Rept. No. 1143. U.S. Navy Dept., Washington, D.C., 1958.

H3a. Hadamard, J. S., *Compt. Rend.* **152**, 1735 (1911); **154**, 109 (1912).

H4. Happel, J., *J. Appl. Phys.* **28**, 1288 (1957).

H5. Happel, J., *A.I.Ch.E.* (*Am. Inst. Chem. Engrs.*) *J.* **4**, 197 (1958).

H6. Happel, J., *Trans. N.Y. Acad. Sci.* [2] **20**, 404 (1958).

H7. Happel, J., *A.I.Ch.E.* (*Am. Inst. Chem. Engrs.*) *J.* **5**, 174 (1959).

H8. Happel, J., and Brenner, H., *A.I.Ch.E.* (*Am. Inst. Chem. Engrs.*) *J.* **3**, 506 (1957).

H9. Happel, J., and Brenner, H., "Low Reynolds Number Hydrodynamics." Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

H9a. Hartunian, R. A., and Liu, S. W., *Phys. Fluids* **6**, 349 (1963).

H10. Hasimoto, H., *J. Fluid Mech.* **5**, 317 (1959).

H11. Hawksley, P. G. W., *Brit. Coal Util. Res. Assoc. Monthly Bull.* **15**, 105 (1951).

H12. Heiss, J. F., and Coull, J., *Chem. Eng. Progr.* **48**, 133 (1952).

H12a. Hermans, J. J., *in* "Flow Properties of Disperse Systems" (J. J. Hermans, ed.), pp. 226–233. Wiley (Interscience), New York, 1953.

H13. Hill, R., and Power, G., *Quart J. Mech. Appl. Math.* **9**, 313 (1956).

H13a. Hocking, L. M., *in* "Aerodynamic Capture of Particles" (E. G. Richardson, ed.), pp. 154–159. Pergamon, New York, 1960.

H14. Hocking, L. M., *J. Fluid Mech.* **20**, 129 (1964).

H15. Hubbard, R. M., and Brown, G. G., *Ind. Eng. Chem.* (*Anal. Edition*) **15**, 212 (1943).

I1. Illingsworth, C. R., *in* "Laminar Boundary Layers" (L. Rosenhead, ed.), p. 163. Oxford Univ. Press, London and New York, 1963.

I2. Illingsworth, C. R., *J. Appl. Math. Phys.* (*ZAMP*) **14**, 681 (1963).

J1. Janke, N. C., Effect of shape upon the settling velocity of regular geometric particles. Ph.D. Dissertation, Univ. California, Los Angeles, California, 1963.

J2. Jayaweera, K. O. L. F., Mason, B. J., and Slack, G. W., *J. Fluid Mech.* **20**, 121 (1964).

J2a. Jayaweera, K. O. L. F., and Mason, B. J., *J. Fluid Mech.* **22**, 709 (1965).

J3. Jeffery, G. B., *Proc. London Math. Soc.* [2] **14**, 327 (1915).

J4. Jeffery, G. B., *Proc. Roy. Soc.* **A102**, 161 (1922).

J5. Jeffrey, R. C., Particle motion in Poiseuille flow. Ph.D. Dissertation, Cambridge Univ., Cambridge, England, 1964; see also Jeffrey, R. C., and Pearson, J. R. A., *J. Fluid Mech.* **22**, 721 (1965).

J6. Jones, A. M., and Knudsen, J. G., *A.I.Ch.E.* (*Am. Inst. Chem. Engrs.*) *J.* **7**, 20 (1961).

K1. Kanwal, R. P., *J. Fluid Mech.* **10**, 17 (1961).

K2. Kanwal, R. P., *J. Fluid Mech.* **19**, 631 (1964).

K3. Kaplun, S., *J. Math. Mech.* **6**, 595 (1957).

K4. Kaplun, S., and Lagerstrom, P. A., *J. Math. Mech.* **6**, 585 (1957).

K4a. Karnis, A., The flow of suspensions through tubes. Ph.D. Dissertation, McGill University, Montreal, Canada, 1966.

K5. Karnis, A., Goldsmith, H. L., and Mason, S. G., *Nature* **200**, 159 (1963).

K5a. Karnis, A., Goldsmith, H. L., and Mason, S. G., The kinetics of flowing dispersions: I. Concentrated suspensions of rigid particles. *J. Coll. Interface Sci.* (in press).

K5b. Karnis, A., Goldsmith, H. L., and Mason, S. G., The flow of suspensions through tubes: V. Inertial effects. *Can. J. Chem. Eng.* (in press).

K5c. Karnis, A., and Mason, S. G., The flow of suspensions through tubes: VI. Meniscus effects. *J. Coll. Interface Sci.* (in press).

K5d. Karnis, A., and Mason, S. G., Particle motions in sheared suspensions: XIX. Viscoelastic media (to be published).

K6. Kaufman, R. N., *J. Appl. Math. Mech.* (*English Transl.*) **27**, 262 (1963); see *Prikl. Mat. Mekh.* **27**, 179 (1963).

K7. Kearsley, E. A., *Arch. Rat. Mech. Anal.* **5**, 347 (1960).

K8. Keller, J. B., *J. Fluid Mech.* **18**, 94 (1964).

K9. Khamrui, S. R., *Bull. Calcutta Math. Soc.* **48**, 159 (1956).

K10. Khamrui, S. R., *Bull. Calcutta Math. Soc.* **52**, 63 (1960).

K10a. Kohlman, D. L., *Mass. Inst. Technol., Fluid Dynamics Res. Lab. Rept.* **63/1** (1963).

K11. Kunkel, W. B., *J. Appl. Phys.* **19**, 1056 (1948).

K12. Kuwabara, S., *J. Phys. Soc. Japan* **14**, 527 (1959).

L1. Ladyzhenskaya, O. A., "The Mathematical Theory of Viscous Incompressible Flow." Gordon & Breach, New York, 1963.

L2. Lagerstrom, P. A., and Cole, J. D., *J. Rat. Mech. Anal.* **4**, 817 (1955).

L3. Lagerstrom, P. A., *J. Math. Mech.* **6**, 605 (1957).

L4. Lagerstrom, P. A., *in* "Theory of Laminar Flows" (F. K. Moore, ed.). Princeton Univ. Press, Princeton, New Jersey, 1964.

L4a. Lagerstrom, P. A., and Chang, I-Dee, *in* "Handbook of Engineering Mechanics" (W. Flügge, ed.), Chapter 81. McGraw-Hill, New York, 1962.

L5. Lamb, H., "Hydrodynamics," 6th ed. Dover, New York, 1945.

L5a. Lanczos, C., "Linear Differential Operators." Van Nostrand, Princeton, New Jersey, 1961.

L5b. Landau, L. D., and Lifshitz, E. M., "Statistical Physics." Addison-Wesley, Reading, Massachusetts, 1958.

L6. Landau, L. D., and Lifshitz, E. M., "Fluid Mechanics," pp. 70–71. Addison-Wesley, Reading, Massachusetts, 1959.

L7. Landau, L. D., and Lifshitz, E. M., "Electrodynamics of Continuous Media," pp. 337–343. Addison-Wesley, Reading, Massachusetts, 1960.

L7a. Landau, L. D., and Lifshitz, E. M., "Mechanics." Addison-Wesley, Reading, Massachusetts, 1960.

L8. Langlois, W. E., *Quart. Appl. Math.* **21**, 61 (1963).

L9. Langlois, W. E., "Slow Viscous Flow." Macmillan, New York, 1964.

L10. Leslie, F. M. (with an appendix by R. I. Tanner), *Quart. J. Mech. Appl. Math.* **14**, 36 (1961).

L10a. Levich, V. G., "Physicochemical Hydrodynamics." Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

L11. Lorentz, H. A., *Koninkl. Ned. Akad. Wetenschap., Verslag.* **5**, 168 (1896); "Collected Papers," Vol. 4, pp. 7–14. Martinus Nijhoff, The Hague, Holland, 1937.

M1. Mackay, G. D. M., and Mason, S. G., *J. Colloid Sci.* **16**, 632 (1961).

M2. Mackay, G. D. M., Suzuki, M., and Mason, S. G., *J. Colloid Sci.* **18**, 103 (1963).

M3. Majumder, S. R., *Indian J. Theoret. Phys.* **8**, 39 (1960).

M4. Malaika, J., Effect of shape of particles on their settling velocity. Ph.D. Dissertation, State Univ. of Iowa, Iowa City, 1949.
M5. Mason, S. G., and Bartok, W., *in* "Rheology of Disperse Systems" (C. C. Mill, ed.), Chapter 2. Pergamon Press, Oxford, 1959.
M5a. Maude, A. D., *Brit. J. Appl. Phys.* **10**, 371 (1959).
M6. Maude, A. D., *Brit. J. Appl. Phys.* **12**, 293 (1961).
M7. Maude, A. D., *Brit. J. Appl. Phys.* **14**, 894 (1963).
M7a. Maude, A. D., and Whitmore, R. L., *Brit. J. Appl. Phys.* **7**, 98 (1956).
M7b. Maxworthy, T., *J. Fluid Mech.* **23**, 369, 373 (1965).
M8. McNown, J. S., *Houille Blanche* **6**, 701 (1951).
M9. McNown, J. S., Analysis of the rolling ball viscometer. Mich. Eng. Coll. Ind. Program, Brochure IP-160. Univ. of Michigan, Ann Arbor, Michigan, 1956.
M10. McNown, J. S., Lee, N. M., McPherson, M.B., and Engez, S. M., *Proc. 7th Intern. Congr. Appl. Mech., London, 1948*, Vol. 2, Part I, p. 17 (1948). (Reprinted as State University of Iowa, Reprints in Engineering, Reprint 81).
M11. McNown, J. S., and Malaika, J., *Trans. Am. Geophys. Union* **31**, 74 (1950).
M12. McNown, J. S., Malaika, J., and Pramanik, H. R., *Trans. 4th Meeting Intern. Assoc. Hydraulic Res. Bombay, India*, p. 511 (1951).
M13. McNown, J. S., and Newlin, J. T., *Proc. 1st Natl. Congr. Appl. Mech., 1951*, p. 801. J. W. Edwards, Ann Arbor, Michigan, 1952.
M14. McPherson, M. B., Boundary influence on the fall velocity of spheres at Reynolds numbers beyond the Stokes range. M.S. Dissertation, State Univ. of Iowa, Iowa City, 1947.
M15. Menzel, D. H., "Mathematical Physics," pp. 86–141. Dover, New York, 1961.
M16. Milne, E. A., "Vectorial Mechanics." Methuen, London, 1948.
M17. Milne-Thomson, L. M., "Theoretical Hydrodynamics," 4th ed., pp. 458–62. Macmillan, New York, 1960.
M18. Mirsky, L., "An Introduction to Linear Algebra." Oxford Univ. Press, London and New York, 1955.
M19. Mohr, C M., and Blumberg, P. N., "Effect of orientation on the settling characteristics of cylindrical particles" (to be published).
M20. Morse, P. M., and Fesbach, H., "Methods of Theoretical Physics," Vols. I and II. McGraw-Hill, New York, 1953.
O1. O'Brien, V., Axi-symmetric viscous flows correct to the first-order in the Reynolds number. Appl. Phys. Lab. Rept. No. CM-1003. Johns Hopkins Univ., Silver Spring, Maryland, 1961.
O1a. O'Brien, V., *Phys. Fluids* **6**, 1356 (1963).
O1b. O'Brien, V., Deformed spheroids in Stokes flow. Appl. Phys. Lab. Tech. Memo TG-716. Johns Hopkins Univ., Silver Spring, Maryland, 1965; see also *A.P.L. Technical Digest* **4**, 11 (1965).
O2. Oliver, D. R., *Nature* **194**, 1269 (1962).
O2a. O'Neill, M. E., *Mathematika* **11**, 67 (1964).
O3. Oseen, C. W., *Arkiv Mat. Astron. Fysik* **6**, No. 29 (1910).
O4. Oseen, C. W., "Neuere Methoden und Ergebnisse in der Hydrodynamik." Akademische Verlags., Leipzig, 1927.
O5. Ovseenko, Yu. G., *Tr. Novocherk. Politekhn. Inst.* **109**, 51 (1960).
O6. Ovseenko, Yu. G., *Izv. Vysshikh Uchebn. Zavedenii, Matematika* **35**, 129 (1963).
P1. Payne, L. E., and Pell, W. H., *J. Fluid Mech.* **7**, 529 (1960).
P2. Pell, W. H., and Payne, L. E., *Mathematika* **7**, 78 (1960).
P3. Pell, W. H., and Payne, L. E., *Quart. Appl. Math.* **18**, 257 (1960).

P4. Pérès, J., *Compt. Rend.* **188**, 310, 440 (1929).
P4a. Perrin, F., *J. Phys. Radium* **7**, 1 (1936).
P5. Perry, J. H., ed., "Chemical Engineers' Handbook," 4th ed., Sect. **19**, p. 15. McGraw-Hill, New York, 1963.
P6. Pettyjohn, E. S., and Christiansen, E. B., *Chem. Eng. Progr.* **44**, 157 (1948).
P7. Pfeffer, R., *Ind. Eng. Chem., Fundamentals* **3**, 380 (1964).
P8. Pfeffer, R., and Happel, J., *A.I.Ch.E. (Am. Inst. Chem. Engrs.) J.* **10**, 605 (1964).
P9. Pliskin, I., and Brenner, H., *J. Fluid Mech.* **17**, 89 (1963).
P10. Polubarinova-Kochina, P. Ya., "Theory of Ground-Water Movement." Princeton Univ. Press, Princeton, New Jersey, 1962.
P11. Proudman, I., and Pearson, J. R. A., *J. Fluid Mech.* **2**, 237 (1957).
P12. Pshenai-Severin, S., *Izv. Akad. nauk SSSR, ser. geofiz.*, 1045 (1957).
P13. Pshenai-Severin, S., *Izv. Akad. nauk SSSR, ser. geofiz.*, 1254 (1958).
R1. Rathna, S. L., *Quart. J. Mech. Appl. Math.* **15**, 427 (1962).
R2. Rayleigh, Lord, *Phil. Mag.* [5] **36**, 354 (1893).
R3. Relton, F. E., *Phil. Mag.* [7] **11**, 129 (1931).
R4. Relton, F. E., *Proc. Roy. Soc.* **A134**, 47 (1931–1932).
R4a. Repetti, R. V., and Leonard, E. F., *Nature* **203**, 1346 (1964).
R4b. Repetti, R. V., and Leonard, E. F., Physical basis for the axial accumulation of red cells. Paper No. 32b presented at 56th National A.I.Ch.E. Meeting, San Francisco, California, 1965.
R5. Richardson, J. G., *in* "Handbook of Fluid Dynamics" (V. L. Streeter, ed.), Chapter 16. McGraw-Hill, New York, 1961.
R5a. Ripps, D. L., and Brenner, H., The Stokes resistance of a slightly deformed sphere: II. Intrinsic resistance operators for an arbitrary initial flow (to be published); see also Ripps, D. L., Invariant differential operators for slow viscous flow past a particle. Ph.D. Dissertation, New York University, New York, 1966.
R5b. Ripps, D. L., and Brenner, H., Slow viscous flow through an infinite periodic lattice of identical solid particles of arbitrary shape (to be published).
R6. Roscoe, R., *Phil. Mag.* [7] **40**, 338 (1949).
R6a. Rubinow, S. I., *Biorheology* **2**, 117 (1964).
R7. Rubinow, S. I., and Keller, J. B., *J. Fluid Mech.* **11**, 447 (1961).
R8. Ruckenstein, E., *Chem. Eng. Sci.* **19**, 131 (1964).
R9. Rumscheidt, F. D., and Mason, S. G., *J. Colloid Sci.* **16**, 238 (1961).
R10. Rybczyński, W., *Bull. Acad. Sci. Cracovie, Ser. A*, pp. 40–46 (1911).
S1. Saffman, P. G., *J. Fluid Mech.* **1**, 540 (1956).
S1a. Saffman, P. G., *J. Fluid Mech.* **22**, 385 (1965).
S2. Sakadi, Z., *Mem. Fac. Eng., Nagoya Univ.* **10**, 42 (1958).
S3. Scheidegger, A. E., "The Physics of Flow Through Porous Media," 2nd ed. Univ. of Toronto Press, Toronto, 1960.
S4. Scheidegger, A. E., *in* "Encyclopedia of Physics: Fluid Dynamics II" (S. Flügge and C. Truesdell, eds.), Vol. 8, Part 2, p. 625. Springer, Berlin, 1963.
S4a. Segré, S., *Trans. 4th Intern. Congr. Rheology, 1963* (A. L. Copley, ed.), Part 4, p. 103. Wiley (Interscience), London and New York, 1965.
S5. Segré, G., and Silberberg, A., *Nature* **189**, 209 (1961).
S6. Segré, G., and Silberberg, A., *J. Fluid Mech.* **14**, 115, 136 (1962).
S7. Segré, G., and Silberberg, A., *J. Colloid Sci.* **18**, 312 (1963).
S8. Selby, T. W., and Hunstad, N. A., The forced-ball viscometer and its application to the characterization of mineral oil systems. Presented at "Symposium on Non-Newtonian Viscometry," sponsored jointly by Research Division III on Flow

Properties, Committee D-2 (ASTM) and Committee on Petroleum Products (A.P.I.), Washington, D.C., 1960.
S9. Shanks, D., *J. Appl. Math. Phys. (ZAMP)* **34,** 1 (1955).
S9a. Shi, Y. Y., *J. Fluid Mech.* **23,** 657 (1965).
S9b. Shizgal, B., Goldsmith, H. L., and Mason, S. G., *Can. J. Chem. Eng.* **43,** 97 (1965).
S10. Slattery, J. C., *Appl. Sci. Res.* **A10,** 286 (1961).
S11. Slattery, J. C., *A.I.Ch.E. (Am. Inst. Chem. Engrs.) J.* **8,** 663 (1962).
S12. Slattery, J. C., and Bird, R. B., *Chem. Eng. Sci.* **16,** 231 (1961).
S13. Small S., Experiments on the lift and drag of spheres suspended in a low Reynolds number Poiseuille flow. Mech. Eng. Dept., Rept. FLD No. 11. Princeton Univ., Princeton, New Jersey, 1963.
S14. Snyder, L. J., and Stewart, W. E., *A.I.Ch.E. (Am. Inst. Chem. Engrs.)* J. **12,** 167 (1966).
S15. Sonshine, R. M., and Brenner, H., The Stokes translation of two or more particles along the axis of an infinitely long circular cylinder, *Appl. Sci. Res., Ser. A* (in press); see also Sonshine, R. M., The Stokes settling of one or more particles along the axis of finite and infinitely long circular cylinders. Ph.D. Dissertation, New York University, New York, 1966.
S16. Sonshine, R. M., Cox, R. G., and Brenner, H., The Stokes translation of a particle of arbitrary shape along the axis of a circular cylinder filled to a finite depth with viscous liquid: I and II. *Appl. Sci. Res., Ser. A* (in press); see also Sonshine, R. M., The Stokes settling of one or more particles along the axis of finite and infinitely long circular cylinders. Ph.D. Dissertation, New York University, New York, 1966.
S17. Srimathi, C. R., and Bhat, G. N., *Brit. J. Appl. Phys.* **16,** 551 (1965).
S18. Starkey, T. V., *Brit. J. Appl. Phys.* **7,** 52 (1956).
S19. Starkey, T. V., Hewlett, V. A., Roberts, J. H. A., and James, R. E., *Brit. J. Appl. Phys.* **12,** 545 (1961).
S20. Stimson, M., and Jeffery, G. B., *Proc. Roy. Soc.* **A111,** 110 (1926).
T1. Takano, M., and Mason, S. G. (to be published).
T1a. Tanner, R. I., *J. Fluid Mech.* **17,** 161 (1963).
T1b. Taylor, G. I., *Proc. Roy. Soc.* **A138,** 41 (1932).
T2. Taylor, G. I., *Proc. Roy. Soc.* **A146,** 501 (1934).
T2a. Taylor, T. D., *Phys. Fluids* **6,** 987 (1963).
T2b. Taylor, T. D., *Intern. J. Heat Mass Transfer* **6,** 993 (1963).
T2c. Taylor, T. D., and Acrivos, A., *J. Fluid Mech.* **18,** 466 (1964).
T3. Tchen, C. M., *J. Appl. Phys.* **25,** 463 (1954).
T3a. Theodore, L., Sidewise force exerted on a spherical particle in a Poiseuillian flow field. Eng. Sc.D. Dissertation, New York University, New York, 1964.
T4. Thomas, R. H., and Walters, K., *Quart. J. Mech. Appl. Math.* **17,** 39 (1964).
T5. Timbrell, V., *Brit. J. Appl. Phys.* **5,** Suppl. 3, S12 (1954).
T6. Tollert, H., *Chem.-Ing.-Tech.* **26,** 141 (1954).
T7. Tomita, Y., *Bull. JSME (Japan. Soc. Mech. Engrs.)* **2,** 469 (1959).
T8. Tuck, E. O., *J. Fluid Mech.* **18,** 619 (1964).
U1. Uchida, S., *Rept. Inst. Sci. Technol., Univ. Tokyo,* **3,** 97 (1949); abstracted in *Ind. Eng. Chem.* **46,** 1194 (1954).
V1. van Dyke, M., "Perturbation Methods in Fluid Mechanics." Academic Press, New York, 1964.
V2. Vand, V., *J. Phys. & Colloid Chem.* **52,** 277 (1948).
V3. Vand, V., *J. Phys. & Colloid Chem.* **52,** 300 (1948).
V4. Villat, H., "Leçons sur les Fluides Visqueux." Gauthier-Villars, Paris, 1943.

W1. Wadhwa, Y. D., *J. Sci. Eng. Res., Indian Inst. Technol., Kharagpur* **2**, 245 (1958).
W2. Wallick, G. C., Savins, J. G., and Arterburn, D. R., *Phys. Fluids* **5**, 367 (1962).
W3. Walters, K., The rotating-sphere elastoviscometer. Presented at 35th Annual Meeting of the Society of Rheology, Mellon Institute, Pittsburgh, Pa., 1964.
W3a. Walters, K., and Water, D., *Rheol. Acta* **3**, 312 (1963–1964).
W3b. Walters, K., and Water, D., *Brit. J. Appl. Phys.* **15**, 989 (1964).
W4. Walters, K., and Waters, N. D., *Brit. J. Appl. Phys.* **14**, 667 (1963).
W5. Wasserman, M. L., and Slattery, J. C., *A.I.Ch.E. (Am. Inst. Chem. Engrs.) J.* **10**, 383 (1964).
W6. Waters, N. D., *Brit. J. Appl. Phys.* **15**, 500 (1964).
W7. Whitehead, A. N., *Quart. J. Math.* **23**, 78, 143 (1889).
W8. Williams, W. E., *Phil. Mag.* [6] **29**, 526 (1915).
W8a. Williams, W. E., *J. Fluid Mech.* **24**, 285 (1966).
W9. Willmarth, W. W., Hawk, N. E., and Harvey, R. L., *Phys. Fluids* **7**, 197 (1964).
Z1. Ziegenhagen, A. J., Bird, R. B., and Johnson, M. W., Jr., *Trans. Soc. Rheol.* **5**, 47 (1961).
Z2. Zierep, J., *Zeit. Flugwiss.* **3**, 22 (1955).